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Unification of different numerical methods for the solution of linear fractional differential equation

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Abstract

In this paper, an attempt has been made to obtain the solution of a linear ordinary fractional differential equation by Generalized Differential Transform Method (GDTM), Homotopy Analysis Method (HAM), Adomian Decomposition Method (ADM) and Power Series Method (PSM). Differential equations of fractional order, as generalization of classical differential equations of integer order, are increasingly used to model problems in fluid flow, finance and other areas of applications. Fractional derivatives are considered in the Caputo sense. Using the present methods we can solve many linear and nonlinear fractional differential equations. These methods perform extremely well in terms of accuracy, efficiency and simplicity and provide an analytical solution in the form of an infinite power series with easily computable components.

Keywords: fractional linear differential equation, generalized differential transform method, homotopy analysis method, adomian decomposition method, power series method, caputo fractional derivative, analytic solution

Introduction

In recent years fractional derivatives have found to be valuable tools in the modeling of any physical phenomena in various fields of science and engineering, finance and hydrology (see Podlubny, 1999) [38]. It has been observed that fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. Zhou (1986) [53] proposed the differential transform method (DTM) to solve linear and nonlinear initial value problems in electric circuit analysis. Differential transform method (DTM) constructs an analytical solution in the form of a polynomial and different from the traditional higher order Taylor series method. This method has been used for solving various types of equations by many authors (see Chen and Ho, 1999; Jang and Chen, 2001; Ayaz, 2004; Arikoglu and Ozkol, 2006; Arikoglu and Ozkol, 2007; Momani *et al*., 2008; Tari *et al*., 2009; Kangalgil and Ayaz, 2009; Abazari and Borhanifar, 2010; Nazari and Shahmorad, 2010; Borhanifar and Abazari, 2010; Soltanalizadeh and Zarebnia, 2011; Borhanifar and Abazari, 2011) [17, 24, 10, 8, 9, 33, 36, 37, 43, 25, 1, 35, 14, 40, 15]. For solving two-dimensional linear and nonlinear partial differential equations of fractional order DTM is further developed as generalized differential transform method (GDTM) (see Momani *et al*., 2007; Odibat and Momani, 2008; Odibat *et al*., 2008) [34, 33, 36, 37]. Recently, Ertiirka and Momanib (2010) [19] applied generalized differential transform method to solve fractional integro-differential equations. Garg *et al*. (2011) [20] implemented generalized differential transform method (GDTM) to derive the solution of space-time fractional telegraph equation. Bansal and Jain (2015) [12] applied generalized differential transform method to solve fractional order Riccati differential equation. A new analytical approach that can be applied to solve nonlinear fractional differential equations is to employ the homotopy analysis method (HAM) (see Liao, 1992, 1995, 1997, 2004, 2009) [26, 27, 28, 29, 30, 52]. Liao (1992) [26] proposed the homotopy analysis method (HAM), is a powerful method to solve the nonlinear problems. Homotopy analysis method (HAM) provides a simple way to adjust and control the convergence of the solution series by introducing a parameter $\hbar$,known as convergence-control parameter. This method has also been successfully applied to solve many types of nonlinear problems by several authers (see Song and Zhang, 2007; Abdulaziz *et al*., 2008; Alomari *et al*., 2008; Xu and Cang, 2008; Hashim *et al*., 2009; Song and Zhang, 2009; Xu *et al*., 2009; Jafari and Seifi, 2009; Jafari and Seifi, 2009) [41, 2, 7, 51, 21, 42, 52, 22, 23]. Adomian (1988, 1994) [3, 4] introduced the adomian decomposition method (ADM), possesses a great potential in solving different kinds of functional equations. Both linear and nonlinear equations and systems of such types are all amenable to the method. This method has the advantage of dealing directly with the problem in case of nonlinear differential equations and nonlinear partial differential equations, and equations are solved without transforming them to more simple ones. The adomian decomposition method avoids linearization, perturbation, discretization, or any unrealistic assumptions. In case of nonlinear equations, the nonlinearity term is replaced by a series of what are called Adomian polynomials and the evaluation of these polynomials is essential, as they contribute to the solution's series components. Thus, relations and algorithms have been deduced and continuously improved to obtain such polynomials in an easy way (see Wazwaz, 2000; Biazar *et al*., 2003; Babolian and Javadi, 2004; Zhu *et al*., 2005; Pourdarvish, 2006) [45, 46, 13, 11, 54, 39].

2. Formation of the problem

Consider the linear ordinary fractional differential equation:

$$\frac{dy(x)}{dx}+\frac{d^{1/2}y(x)}{dx^{1/2}}+y(x)=x\ ;\quad x\in[0,1] \tag{1}$$

subject to the initial conditions $y(0)=1$ and $y'(0)=-1$ (2)

where $\frac{d^{1/2}}{dx^{1/2}}$ is the fractional differential operator(Caputo derivative) of order $1/2$.

3. Mathematical Preliminaries on Fractional Calculus

In the present analysis we introduce the following definitions (see Miller and Ross, 1993; Das, 2008) [32, 18].

3.1 Definition: Let $\alpha\in R^+$. On the usual Lebesgue space $L(a,b)$ integral operator I^α defined by

$$I^\alpha f(x)=\frac{d^{-\alpha}f(x)}{dx^{-\alpha}}=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)\,dt$$ and $I^0 f(x)=f(x)$ is called Riemann-Liouville fractional integral operator of order $\alpha\geq 0$ and $a\leq x<b$

It has the following properties:

$I^\alpha f(x)$ exists for any $x\in[a,b]$

$$I^\alpha I^\beta f(x)=I^{\alpha+\beta}f(x)$$

$$I^\alpha I^\beta f(x)=I^\beta I^\alpha f(x)$$

$$I^\alpha x^\gamma=\frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}x^{\alpha+\gamma}$$

Where $f(x)\in L[a,b]$, $\alpha,\beta\geq 0$, $\gamma>-1$

3.2 Definition: The Riemann-Liouville definition of fractional order derivative is

$${}_0^{RL}D_x^\alpha f(x)=\frac{d^n}{dx^n}\,{}_0I_x^{n-\alpha}f(x)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x (x-t)^{n-\alpha-1}f(t)\,dt,$$

where n is an integer that satisfies $n-1<\alpha<n$.

3.3 Definition: A modified fractional differential operator ${}_0^{c}D_x^\alpha$ proposed by Caputo is given by

$${}_0^{c}D_x^\alpha f(x)={}_0I_x^{n-\alpha}\frac{d^n}{dx^n}f(x)=\frac{1}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1}f^{(n)}(t)\,dt,$$

Where $\alpha(\alpha\in R^+)$ is the order of operation and n is an integer that satisfies $n-1<\alpha<n$.

It has the following two basic properties (see Caputo, 1967):

If $f \in L_\infty(a,b)$ or $f \in C[a,b]$ and $\alpha > 0$ then ${}_0^cD_x^\alpha\, {}_0I_x^\alpha f(x) = f(x)$.

If $f \in C^n[a,b]$ and if $\alpha > 0$ then

$${}_0I_x^\alpha\, {}_0^cD_x^\alpha f(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0^+)}{k!} x^k \; ; \; n-1 < \alpha < n .$$

3.4 Definition: For m being the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order $\alpha > 0$, is defined as (see Podlubny, 1999) [38]

$$D_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \dfrac{\partial^m u(x,\xi)}{\partial \xi^m} & ; \quad \alpha = m \in N \\ \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t (t-\xi)^{m-\alpha-1} \frac{\partial^m u(x,\xi)}{\partial \xi^m} d\xi & ; \quad m-1 \le \alpha < m \end{cases}$$

Relation between Caputo derivative and Riemann-Liouville derivative:

$${}_0^cD_x^\alpha f(x) = {}_0^{RL}D_t^\alpha f(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(0^+)}{\Gamma(k-\alpha+1)} x^{k-\alpha} \; ; m-1 < \alpha < m$$

Integrating by parts, we get the following formulae as given by (see Almeida and Torres, 2011)

$$\int_a^b g(x)\, {}_a^cD_x^\alpha f(x) dx = \int_a^b f(x)\, {}_x^{RL}D_b^\alpha g(x) dx + \sum_{j=0}^{n-1} \left[{}_x^{RL}D_b^{\alpha+j-n} g(x)\, {}_x^{RL}D_b^{n-j-1} f(x) \right]_a^b$$

For $n=1$, $\int_a^b g(x)\, {}_a^cD_x^\alpha f(x) dx = \int_a^b f(x)\, {}_x^{RL}D_b^\alpha g(x) dx + \left[{}_xI_b^{1-\alpha} g(x).f(x) \right]_a^b$

4. Generalized one dimensional differential transform method:

Generalized differential transform of a function $y(x)$ in one variable is denoted by $Y_\alpha(k)$ and defined as follows (see Momani *et al*., 2007; Odibat and Momani, 2008; Odibat *et al*., 2008; Ertiirka and Momanib, 2010; Garg *et al*., 2011; Bansal and Jain, 2015) [34, 33, 36, 37, 20, 12]:

$$Y_\alpha(k) = \frac{1}{\Gamma(\alpha k+1)} \left[\left(D_{x_0}^\alpha \right)^k y(x) \right]_{x=x_0} \tag{3}$$

where $\alpha \in (0,1]$ and $\left(D_{x_0}^\alpha \right)^k = D_{x_0}^\alpha, D_{x_0}^\alpha, \ldots.., D_{x_0}^\alpha$ ($k-$ times).

and the inverse generalized differential transform of $Y_\alpha(k)$ is given by

$$y(x) = \sum_{k=0}^{\infty} Y_\alpha(k) (x-x_0)^{\alpha k} \tag{4}$$

It has the following properties:

If $u(x)=v(x)\pm w(x)$ then $U_\alpha(k)=V_\alpha(k)\pm W_\alpha(k)$

If $u(x)=av(x)$; $a\in R$ then $U_\alpha(k)=aV_\alpha(k)$

If $U(x)=v(x)w(x)$ then $U_\alpha(x)=\sum_{r=0}^{k}V_\alpha(r)W_\alpha(k-r)$

If $u(x)=(x-x_0)^{n\alpha}$ then $U_\alpha(k)=\delta(k-n)$

If $u(x)=D_{x_0}^{\alpha}v(x)$; $0<\alpha\le 1$ then $U_\alpha(k)=\dfrac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)}V_\alpha(k+1)$

I f $u(x)=x^\lambda f(x)$ where $\lambda>-1$, $f(x)$ has the generalized Taylor series expansion $f(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^{n\alpha}$ and $\beta<\lambda+1$ and α is arbitrary or

$\beta\ge\lambda+1$, α arbitrary and $a_n=0$ for $n=0,1,2,....m-1$, where $m-1<\beta\le m$ .Then (3) becomes

$$U_\alpha(k)=\frac{1}{\Gamma(\alpha k+1)}\left[D_{x_0}^{\alpha k}u(x)\right]_{x_0}$$

If $u(x)=D_{x_0}^{\gamma}f(x)$, $m-1<\gamma\le m$ and the function $f(x)$ satisfies the conditions given in (VI) then $U_\alpha(k)=\dfrac{\Gamma(\alpha k+\gamma+1)}{\Gamma(\alpha k+1)}F_\alpha\left(k+\dfrac{\gamma}{\alpha}\right)$

where $U_\alpha(k)$, $V_\alpha(k)$, $W_\alpha(k)$ and $F_\alpha(k)$ are the differential transformations of the functions $u(x),v(x),w(x)$ and $f(x)$ respectively and

$$\delta(k-n)=\begin{cases}1 & ; \quad k=n\\ 0 & ; \quad k\ne n\end{cases}$$

Applying generalized differential transform (3) with $x_0=0$ on (1) and (2) we obtain

$$Y_{1/2}(k+2)=\frac{\Gamma\left(\frac{k}{2}+1\right)}{\Gamma\left(\frac{k}{2}+2\right)}\left[\delta(k-2)-Y_{1/2}(k)-\frac{\Gamma\left(\frac{k}{2}+\frac{3}{2}\right)}{\Gamma\left(\frac{k}{2}+1\right)}Y_{1/2}(k+1)\right] \tag{5}$$

and $$Y_{\frac{1}{2}}(0)=1, Y_{\frac{1}{2}}(1)=0, Y_{\frac{1}{2}}(2)=-1 \qquad (6)$$

Now utilizing the recurrence relation (5) and the initial conditions (6), we obtain after a little simplification the following values of $Y_{\frac{1}{2}}(k)$ for $k=0,1,2,......$

$$Y_{\frac{1}{2}}(3)=\frac{1}{\Gamma\left(\frac{5}{2}\right)},\quad Y_{\frac{1}{2}}(4)=\frac{1}{\Gamma(3)},\quad Y_{\frac{1}{2}}(5)=-\frac{2}{\Gamma\left(\frac{7}{2}\right)},\quad Y_{\frac{1}{2}}(6)=\frac{1}{\Gamma(4)},\quad Y_{\frac{1}{2}}(7)=\frac{1}{\Gamma\left(\frac{9}{2}\right)},$$

$$Y_{\frac{1}{2}}(8)=-\frac{2}{\Gamma(5)},\quad Y_{\frac{1}{2}}(9)=\frac{1}{\Gamma\left(\frac{11}{2}\right)},\quad Y_{\frac{1}{2}}(10)=\frac{1}{\Gamma(6)},\quad Y_{\frac{1}{2}}(11)=-\frac{2}{\Gamma\left(\frac{13}{2}\right)},\quad Y_{\frac{1}{2}}(12)=\frac{1}{\Gamma(7)},$$

$$Y_{\frac{1}{2}}(13)=\frac{1}{\Gamma\left(\frac{15}{2}\right)},\quad Y_{\frac{1}{2}}(14)=-\frac{2}{\Gamma(8)},\quad Y_{\frac{1}{2}}(15)=\frac{1}{\Gamma\left(\frac{17}{2}\right)},\quad Y_{\frac{1}{2}}(16)=\frac{1}{\Gamma(9)},\quad Y_{\frac{1}{2}}(17)=-\frac{2}{\Gamma\left(\frac{19}{2}\right)}$$

and so on Now, from (4), we have

$$y(x)=\sum_{k=0}^{\infty} Y_{\frac{1}{2}}(k)\, x^{\frac{k}{2}} \qquad (7)$$

Using the above values of $Y_{\frac{1}{2}}(k)$; $k=0,1,2,.....$ in (4) the solution of (1) is obtained as

$$\begin{aligned}
y(x)&=1-x+\frac{1}{\Gamma(3)}x^{2}+\frac{1}{\Gamma(4)}x^{3}-\frac{2}{\Gamma(5)}x^{4}+\frac{1}{\Gamma(6)}x^{5}+\frac{1}{\Gamma(7)}x^{6}-\frac{2}{\Gamma(8)}x^{7}\\
&+\frac{1}{\Gamma(9)}x^{8}+\frac{1}{\Gamma\left(\frac{5}{2}\right)}x^{\frac{3}{2}}-\frac{2}{\Gamma\left(\frac{7}{2}\right)}x^{\frac{5}{2}}+\frac{1}{\Gamma\left(\frac{9}{2}\right)}x^{\frac{7}{2}}+\frac{1}{\Gamma\left(\frac{11}{2}\right)}x^{\frac{9}{2}}-\frac{2}{\Gamma\left(\frac{13}{2}\right)}x^{\frac{11}{2}}\\
&+\frac{1}{\Gamma\left(\frac{15}{2}\right)}x^{\frac{13}{2}}+\frac{1}{\Gamma\left(\frac{17}{2}\right)}x^{\frac{15}{2}}-\frac{2}{\Gamma\left(\frac{19}{2}\right)}x^{\frac{17}{2}}+.......
\end{aligned} \qquad (8)$$

5. Homotopy Analysis Method (HAM):

Let us consider the fractional differential equation

$$\aleph\left(u(x)\right)=0$$

Where $\aleph$ is a nonlinear operator, x is independent variable, $u(x)$ is an unknown function, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao (2004, 2012) [29, 31] constructs the so-called zero-order deformation equation

$$(1-q)L\left[\phi(x;q)-u_{0}(x)\right]=q\hbar\aleph\left(\phi(x;q)\right) \qquad (9)$$

Here $q\in[0,1]$ is the embedding parameter, $\hbar\neq 0$ is a convergence-control parameter to be chosen later on, L is an auxiliary linear differential operator of integer order and possesses the property $L(C)=0$, where C and $\phi=\phi(x;q)$ are constant and unknown function respectively.

Expanding $\phi(x;q)$ in Taylor series with respect to q ,we have

$$\phi(x;q) = u_0(x) + \sum_{m=1}^{\infty} u_m(x) q^m \tag{10}$$

where $u_m(x) = \frac{1}{m!}\left[\frac{\partial^m \phi(x;q)}{\partial q^m}\right]_{q=0}$

Differentiating the equation (9) m -times with respect to the embedding parameter q and then setting $q=0$, and finally dividing them by $m!$, we have the m^{th} -order deformation equation

$$L\left[u_m(x) - \lambda_m u_{m-1}(x)\right] = \hbar R_m\left[\vec{u}_{m-1}(x)\right] \tag{11}$$

Where $R_m\left[\vec{u}_{m-1}(x)\right] = \frac{1}{(m-1)!}\left[\frac{\partial^{m-1}\aleph(\phi(x;q))}{\partial q^{m-1}}\right]_{q=0}$, $\lambda_m = \begin{cases} 0 & ; \quad m \le 1 \\ 1 & ; \quad m \ge 2 \end{cases}$

and $\vec{u}_{m-1}(x) = \left[u_0(x), u_1(x), u_2(x), u_3(x), ..., u_{m-1}(x)\right]$

We define the equation (1), in terms of linear and nonlinear operators as follows:

$$L\left[\phi(x;q)\right] = D\phi(x;q) \tag{12}$$

and $\aleph\left[\phi(x;q)\right] = D\phi(x;q) + D_c^{1/2}\phi(x;q) + \phi(x;q) - x$ (13)

where $D = \frac{d}{dx}$

Using (9), we construct the zeroth order deformation equation as

$$(1-q)L\left[\phi(x;q) - y_0(x)\right] = q\hbar\aleph\left[\phi(x;q)\right] \tag{14}$$

Obviously, when $q=0$ and $q=1$ respectively, we can write from equation (14) the following relations:

$\phi(x;0) = y_0(x) = y(0)$ and $\phi(x;1) = y(x)$ (15)

According to equations (10) and (11), we obtain the m^{th} order deformation equation as

$$L\left[y_m(x) - \lambda_m y_{m-1}(x)\right] = \hbar R_m\left[\vec{y}_{m-1}(x)\right] \tag{16}$$

where

$$R_m\left[\vec{y}_{m-1}(x)\right]=\frac{1}{(m-1)!}\left[\frac{\partial^{m-1}\aleph\left[\phi(x;q)\right]}{\partial q^{m-1}}\right]_{q=0}$$

$$=\frac{1}{(m-1)!}\left[\frac{\partial^{m-1}}{\partial q^{m-1}}\left\{D\phi(x;q)+D_c^{1/2}\phi(x;q)+\phi(x;q)-x\right\}\right]_{q=0}$$

$$=Dy_{m-1}(x)+D_c^{1/2}y_{m-1}(x)+y_{m-1}(x)-x(1-\lambda_m) \qquad (17)$$

From equations (16) and (17), we can write

$$L\left[y_m(x)-\lambda_m y_{m-1}(x)\right]=\hbar\left[Dy_{m-1}(x)+D_c^{1/2}y_{m-1}(x)+y_{m-1}(x)-x(1-\lambda_m)\right] \qquad (18)$$

For $m=1$, equation (18) becomes

$$L\left[y_1(x)\right]=\hbar\left[Dy_0(x)+D_c^{1/2}y_0(x)+y_0(x)-x\right] \qquad (19)$$

For $m\geq 2$, equation (18) reduces to

$$y_m(x)=y_{m-1}(x)+\hbar I\left[Dy_{m-1}(x)+D_c^{1/2}y_{m-1}(x)+y_{m-1}(x)\right] \qquad (20)$$

Taking $\hbar=-1$, we get from equations (1),(15), (19) and (20), after some rigorous calculations,

$$y_0(x)=y(0)=1, \quad y_1(x)=-x+\frac{1}{2}x^2, \quad y_2(x)=\frac{1}{2}x^2-\frac{1}{6}x^3+\frac{1}{\Gamma\left(\frac{5}{2}\right)}x^{3/2}-\frac{1}{\Gamma\left(7/2\right)}x^{5/2},$$

$$y_3(x)=-\frac{1}{2}x^2-\frac{1}{\Gamma\left(7/2\right)}x^{5/2}+\frac{1}{3\Gamma\left(9/2\right)}x^{7/2}+\frac{1}{24}x^4,$$

$$y_4(x)=\frac{1}{\Gamma\left(7/2\right)}x^{5/2}+\frac{3}{\Gamma(4)}x^3-\frac{2}{\Gamma(5)}x^4+\frac{2}{\Gamma\left(9/2\right)}x^{7/2}-\frac{3}{\Gamma\left(11/2\right)}x^{9/2}-\frac{1}{\Gamma(6)}x^5,$$

$$y_5(x)=-\frac{1}{\Gamma(4)}x^3-\frac{5}{\Gamma(5)}x^4-\frac{4}{\Gamma\left(9/2\right)}x^{7/2}+\frac{5}{\Gamma(6)}x^5+\frac{4}{\Gamma\left(13/2\right)}x^{11/2}+\frac{1}{\Gamma(7)}x^6,$$

$$y_6(x)=\frac{1}{\Gamma\left(9/2\right)}x^{7/2}+\frac{9}{\Gamma\left(11/2\right)}x^{9/2}+\frac{5}{\Gamma(5)}x^4+\frac{5}{\Gamma(6)}x^5-\frac{5}{\Gamma\left(13/2\right)}x^{11/2}$$

$$-\frac{9}{\Gamma(7)}x^6-\frac{5}{\Gamma\left(15/2\right)}x^{13/2}-\frac{1}{\Gamma(8)}x^7,$$

$$y_7(x) = -\frac{1}{\Gamma(5)}x^4 - \frac{14}{\Gamma(6)}x^5 - \frac{6}{\Gamma\left(11/2\right)}x^{9/2} - \frac{14}{\Gamma\left(13/2\right)}x^{11/2} + \frac{14}{\Gamma\left(15/2\right)}x^{13/2} + \frac{14}{\Gamma(8)}x^7$$

$$+\frac{6}{\Gamma\left(17/2\right)}x^{15/2} + \frac{1}{\Gamma(9)}x^8$$

and so on.
By using (10), we obtain

$$y(x) = y_0(x) + \sum_{m=1}^{\infty} y_m(x) \text{ ; for } q=1 \tag{21}$$

Using the above values of $y_n(x)$; $n = 0, 1, 2, 3, ...$ in (21**),** the solution of (1) is obtained as

$$y(x) = 1 - x + \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{2}{24}x^4 - \frac{5}{\Gamma(6)}x^5 - \frac{8}{\Gamma(7)}x^6 + \frac{13}{\Gamma(8)}x^7 + \frac{1}{\Gamma(9)}x^8 + \frac{1}{\Gamma\left(5/2\right)}x^{3/2}$$

$$-\frac{1}{\Gamma\left(7/2\right)}x^{5/2} - \frac{2}{3\Gamma\left(9/2\right)}x^{7/2} - \frac{15}{\Gamma\left(13/2\right)}x^{11/2} + \frac{9}{\Gamma\left(15/2\right)}x^{13/2} + \frac{6}{\Gamma\left(17/2\right)}x^{15/2} + ... \tag{22}$$

6. Adomian Decomposition Method (ADM):

Let us consider the differential equation in the following form (see Adomian,1994, 1988; Wazwaz, 1999, 2001, 2002, 2005) [4, 3, 44, 47, 48, 49, 50]:

$$L\left[y(x)\right] + N\left[y(x)\right] = f(x)$$

where L is a linear differential operator, N is a nonlinear operator, $f(x)$ is a known function of x, respectively. Adomian decomposition method (ADM) defines the unknown function $y(x)$ by an infinite series (see Adomian, 1994, 1988; Wazwaz, 1999, 2001, 2002, 2005) [4, 3, 44, 47, 48, 49, 50]:

$$y(x) = \sum_{i=0}^{\infty} y_i(x) \tag{23}$$

where the components $y_i(x)$ are determined by using recurrence relations.

The nonlinear part $N\left[y(x)\right]$ can be decomposed into an infinite series of adomian polynomials given by (see Adomian, 1994, 1988; Wazwaz, 1999, 2001, 2002, 2005) [4, 3, 44, 47, 48, 49, 50]

$$N\left[y(x)\right] = \sum_{i=0}^{\infty} A_i\left(y_1, y_2, y_3, ..., y_i\right) \tag{24}$$

where A_i's are the so called adomian polynomials and defined by

$$A_i\left(y_1,y_2,y_3,....,y_i\right)=\frac{1}{i!}\left\{\frac{d^i}{d\lambda^i}\left[N\left(\sum_{j=0}^{i}\lambda^j y_j\left(x\right)\right)\right]\right\}_{\lambda=0}, \text{ for } i=0,1,2,.... \tag{25}$$

Taking $L=\frac{d}{dx}$ and $N\left[y\left(x\right)\right]=D_c^{1/2}y\left(x\right)+y\left(x\right)$,let us rewrite the equation (1) as

$$L\left[y\left(x\right)\right]=-N\left[y\left(x\right)\right]+x \tag{26}$$

Applying L^{-1} on both sides of equation (26), we get

$$y\left(x\right)=\left(y\left(0\right)+\frac{1}{2}x^2\right)-L^{-1}\left\{N\left[y\left(x\right)\right]\right\} \tag{27}$$

Let us choose

$$y_0\left(x\right)=y\left(0\right)+\frac{1}{2}x^2=1+\frac{1}{2}x^2 \tag{28}$$

Then after using equations (23) and (24) in (27), we get

$$y_{i+1}\left(x\right)=-L^{-1}\left[A_i\left(y_0,y_1,y_2,....,y_i\right)\right], \text{ for } i=0,1,2,....$$

where A_i 's are given by (25) and consequently, we obtain

$$y_1\left(x\right)=-\frac{x^3}{\Gamma\left(4\right)}-x-\frac{1}{\Gamma\left(7/2\right)}x^{5/2},$$

$$y_2\left(x\right)=\frac{1}{\Gamma\left(4\right)}x^3+\frac{1}{\Gamma\left(5/2\right)}x^{3/2}+\frac{2}{\Gamma\left(9/2\right)}x^{7/2}+\frac{1}{\Gamma\left(3\right)}x^2+\frac{1}{\Gamma\left(5\right)}x^4,$$

$$y_3\left(x\right)=-\left[\frac{1}{\Gamma\left(9/2\right)}x^{7/2}+\frac{3}{\Gamma\left(5\right)}x^4+\frac{2}{\Gamma\left(7/2\right)}x^{5/2}+\frac{3}{\Gamma\left(11/2\right)}x^{9/2}+\frac{1}{\Gamma\left(3\right)}x^2\right.$$
$$\left.+\frac{1}{\Gamma\left(4\right)}x^3+\frac{1}{\Gamma\left(6\right)}x^5\right],$$

$$y_4\left(x\right)=\frac{2}{\Gamma\left(5\right)}x^4+\frac{3}{\Gamma\left(4\right)}x^3+\frac{6}{\Gamma\left(6\right)}x^5+\frac{1}{\Gamma\left(7\right)}x^6+\frac{4}{\Gamma\left(11/2\right)}x^{9/2}+\frac{1}{\Gamma\left(7/2\right)}x^{5/2}$$
$$+\frac{3}{\Gamma\left(9/2\right)}x^{7/2}+\frac{4}{\Gamma\left(13/2\right)}x^{11/2},$$

$$y_5(x) = -\left[\frac{1}{\Gamma(4)}x^3 + \frac{6}{\Gamma(5)}x^4 + \frac{6}{\Gamma(6)}x^5 + \frac{10}{\Gamma(7)}x^6 + \frac{1}{\Gamma(8)}x^7 + \frac{4}{\Gamma\left(9/2\right)}x^{7/2} + \frac{5}{\Gamma\left(11/2\right)}x^{9/2}\right.$$
$$\left. + \frac{10}{\Gamma\left(13/2\right)}x^{11/2} + \frac{5}{\Gamma\left(15/2\right)}x^{13/2}\right],$$

$$y_6(x) = \frac{5}{\Gamma(5)}x^4 + \frac{11}{\Gamma(6)}x^5 + \frac{16}{\Gamma(7)}x^6 + \frac{15}{\Gamma(8)}x^7 + \frac{1}{\Gamma(9)}x^8 + \frac{1}{\Gamma\left(9/2\right)}x^{7/2}$$
$$+ \frac{10}{\Gamma\left(11/2\right)}x^{9/2} + \frac{11}{\Gamma\left(13/2\right)}x^{11/2} + \frac{20}{\Gamma\left(15/2\right)}x^{13/2} + \frac{6}{\Gamma\left(17/2\right)}x^{15/2},$$

$$y_7(x) = -\left[\frac{1}{\Gamma(5)}x^4 + \frac{15}{\Gamma(6)}x^5 + \frac{22}{\Gamma(7)}x^6 + \frac{36}{\Gamma(8)}x^7 + \frac{21}{\Gamma(9)}x^8 + \frac{1}{\Gamma(10)}x^9\right.$$
$$\left. + \frac{6}{\Gamma\left(11/2\right)}x^{9/2} + \frac{21}{\Gamma\left(13/2\right)}x^{11/2} + \frac{27}{\Gamma\left(15/2\right)}x^{13/2} + \frac{35}{\Gamma\left(17/2\right)}x^{15/2} + \frac{7}{\Gamma\left(19/2\right)}x^{17/2}\right]$$

and so on

By using (23) and above values of $y_n(x)$; $n = 0,1,2,\ldots\ldots$ the solution of (1) is obtained as

$$y(x) = 1 - x + \frac{1}{2}x^2 + \frac{1}{\Gamma(4)}x^3 - \frac{2}{\Gamma(5)}x^4 - \frac{5}{\Gamma(6)}x^5 - \frac{15}{\Gamma(7)}x^6 - \frac{22}{\Gamma(8)}x^7 - \frac{20}{\Gamma(9)}x^8$$
$$- \frac{1}{\Gamma(10)}x^9 + \frac{1}{\Gamma\left(5/2\right)}x^{3/2} - \frac{2}{\Gamma\left(7/2\right)}x^{5/2} + \frac{1}{\Gamma\left(9/2\right)}x^{7/2} - \frac{16}{\Gamma\left(13/2\right)}x^{11/2} - \frac{12}{\Gamma\left(15/2\right)}x^{13/2}$$
$$- \frac{29}{\Gamma\left(17/2\right)}x^{15/2} - \frac{7}{\Gamma\left(19/2\right)}x^{17/2} + \ldots\ldots\ldots\ldots \tag{29}$$

7. Solution of equation (1) by the Power Series Method (PSM):

Let $$y(x) = \sum_{n=0}^{\infty} a_n x^{n/2} \tag{30}$$

be the power series solution of the fractional linear ordinary differential equation (1).

Applying $I^{1/2}$ on both sides of (1), we obtain

$$I^{1/2}\left(Dy(x)\right) + I^{1/2}\left(D_c^{1/2}y(x)\right) + I^{1/2}\left(y(x)\right) = I^{1/2}(x) \tag{31}$$

Now substituting (30) in (31), one gets

$$\sum_{n=1}^{\infty}\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}a_n x^{\frac{n}{2}-\frac{1}{2}}+\sum_{n=1}^{\infty}a_n x^{\frac{n}{2}}+\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n}{2}+\frac{3}{2}\right)}a_n x^{\frac{n}{2}+\frac{1}{2}}=\frac{x^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)} \tag{32}$$

With $a_0 = 1$, equating the co-efficient of different powers of x, we obtain

$$a_1 = 0,\ a_2 = -1,\ a_3 = \frac{1}{\Gamma\left(5/2\right)},\ a_4 = \frac{1}{2},\ a_5 = -\frac{2}{\Gamma\left(7/2\right)},\ a_6 = \frac{1}{6},\ a_7 = \frac{1}{\Gamma\left(9/2\right)},\ a_8 = -\frac{2}{\Gamma(5)},$$
$$a_9 = \frac{1}{\Gamma\left(11/2\right)},\ a_{10} = \frac{1}{\Gamma(6)}$$

and so on.
Then the Power Series Solution of (1) is

$$y(x) = 1 - x + \frac{1}{2}x^2 + \frac{1}{\Gamma(4)}x^3 - \frac{2}{\Gamma(5)}x^4 + \frac{1}{\Gamma(6)}x^5 + \frac{1}{\Gamma\left(5/2\right)}x^{3/2} - \frac{2}{\Gamma\left(7/2\right)}x^{5/2}$$
$$+\frac{1}{\Gamma\left(9/2\right)}x^{7/2} + \frac{1}{\Gamma\left(11/2\right)}x^{9/2} + \ldots\ldots\ldots \tag{33}$$

8. Conclusion

In this paper, four different methods- generalized differential transform method (GDMT), homotopy analysis method(HAM), adomian decomposition method (ADM) and power series method (PSM) applied to find the approximate solution of a linear fractional differential equation. It has been found that the solution obtained by these methods is significantly identical. The advantages of these global methodologies – GDTM, HAM and ADM lie in the fact that they not only lead to an analytical continuous approximation which is very rapidly convergent, but also show the dependence, giving insight into the character and behavior of the solution just as in a closed form solution. Furthermore, these methods do not require any transformation technique, linearization or discretization of the variables and they do not make closure approximation or smallness assumption. These techniques may also be applied to the solution of the nonlinear ordinary differential equations, nonlinear partial differential equations and nonlinear integral equations.

References

1. Abazari R, Borhanifar A. Numerical study of the solution of the Burgers and coupled Burgers equations by a differential transformation method. Comput. Math. Appl. 2010; 59:2711-2722.
2. Abdulaziz O, Hashim I, Saif A. Series solutions of time-fractional PDEs by homotopy analysis method. Differ. Equ. Nonlinear Mech. 2008. DOI:10.1155/2008/686512
3. Adomian G. A review of the decomposition method in applied mathematics, J. Math. Anal. Appl. 1988; 135:501-544.
4. Adomian G. Solving Frontier Problems of Physics: The Decomposition Method. Kluwer Academic Publishers, Dordrecht. 1994.
5. Al-Hayani W, Casasus L. The Adomian decomposition method in turning point problems. J. Comput. and Appl. Math. 2005; 177:187-203.
6. Almeida R, Torres DF. Necessary and sufficient conditions for the fractional calculus of variations with caputo derivatives. Communications in Nonlinear Science and Numerical Simulation. 2011; 16:1490-1500.
7. Alomari AK, Noorani MSM, Nazar RM. Approximate analytical solutions of the Klein-Gordon equation by means of the homotopy analysis method. JQMA. 2008; 4(1):45-57.
8. Arikoglu A, Ozkol I. Solution of difference equations by using differential transformation method. Appl. Math. Comput. 2006; 174:1216-1228.
9. Arikoglu A, Ozkol I. Solution of fractional differential equations by using differential transformation method. Chaos Solitons Fractals. 2007; 34:1473-1481.
10. Ayaz F. Solutions of the systems of differential equations by differential transform Method. Appl. Math. Comput. 2004; 147:547-567.

11. Babolian E, Javadi S. New method for calculating Adomian polynomials, Appl. Math. Comput. 2004; 153:253-259.
12. Bansal MK, Jain R. Application of generalized differential transform method to fractional order Riccati differential equation and numerical results. Int. J. of Pure and Appl. Math. 2015; 99(3):355-366.
13. Biazar J, Babolian E, Kember G, Nouri A, Islam R. An alternate algorithm for computing Adomian polynomials in special cases, Appl. Math. Comput. 2003; 138:523-529.
14. Borhanifar A, Abazari R. Numerical study of nonlinear Schr. dinger and coupled Schr. dinger equations by differential transformation method. Optics Communications. 2010; 283:2026-2031.
15. Borhanifar A, Abazari R. Exact solutions for non-linear Schr. dinger equations by differential transformation method. J. Appl. Math. Comput. 2011; 35:37-51.
16. Caputo M. Linear models of dissipation whose q is almost frequency independent-ii. Geophys J. R. Astron. Soc. 1967; 13:529-539.
17. Chen CK, Ho SH. Solving partial differential equations by two dimensional differential transform method. Appl. Math. Comput. 1999; 106:171-179.
18. Das S. Functional Fractional Calculus. Springer. 2008.
19. Ertiirka VS, Momanib S. On the generalized differential transform method : application to fractional integro-differential equations. Studies in Nonlinear Sciences. 2010; 1(4):118-126.
20. Garg M, Manohar P, Kalla SL. Generalized differential transform method to Space-time fractional telegraph equation. Int. J. of Differential Equations, Hindawi Publishing Corporation, article id.:548982,9 pages,doi.:10.1155/2011/548982.
21. Hashim I, Abdulaziz O, Momani S. Homotopy analysis method for fractional IVPs. Commun. Nonlinear Sci. Numer.Simul. 2009; 14(3):674-684.
22. Jafari H, Seifi S. Homotopy analysis method for solving linear and nonlinear fractional diffusion-wave equation. Commun. Nonlinear Sci. Numer. Simul. 2009; 14(5):2006-2012.
23. Jafari H, Seifi S. Solving a system of nonlinear fractional partial differential equations using homotopy analysis method. Commun. Nonlinear Sci. Numer. Simul. 2009;14(5):1962-1969.
24. Jang MJ, Chen CK. Two-dimensional differential transformation method for partial differantial equations, Appl. Math. Comput. 2001; 121:261-270.
25. Kangalgil F, Ayaz F. Solitary wave solutions for the KDV and mKDV equations by differential transformation method, Choas Solitons Fractals. 2009; 41:464-472.
26. Liao SJ. The proposed homotopy analysis technique for the solution of nonlinear problems. PhD thesis, Shanghai Jiao Tong University, Shanghai, China. 1992.
27. Liao SJ. An approximate solution technique not depending on small parameters: a special example. Int. J.Nonlinear Mech. 1995; 30(3):371-380.
28. Liao SJ. A kind of approximate solution technique which does not depend upon small parameters - II. An application in fluid mechanics. Int. J. Nonlinear Mech. 1997; 32(5):815-822.
29. Liao SJ. Beyond Perturbation: Introduction to the Homotopy Analysis Method. CRC Series: Modern Mechanics and Mathematics, Chapman & Hall/CRC, Boca Raton. 2004, 2.
30. Liao SJ. Notes on the homotopy analysis method: some definitions and theorems. Commun. Nonlinear Sci. Numer. Simul. 2009; 14(4):983-997.
31. Liao SJ. Homotopy Analysis Method in Nonlinear Differential Equations. Higher Education Press, Beijing and Springer Verlag Berlin Heidelberg. 2012.
32. Miller KS, Ross B. An Introduction to the Fractional Calculus and Fractional Differential Equations. John Wiley and Son. 1993.
33. Momani S, Odibat Z, Hashim I. Algorithms for nonlinear fractional partial differantial equations: A selection of numerical methods. Topol. Method Nonlinear Anal. 2008; 31:211-226.
34. Momani S, Odibat Z, Erturk VS. Generalized differential transform method for solving a space- and time-fractional diffusion-wave equation. Physics Letters. A. 2007; 370(5-6):379-387.
35. Nazari D, Shahmorad S. Application of the fractional differential transform method to fractional-order integro-differential equations with nonlocal boundary conditions. J. Comput. Appl. Math. 2010; 234:883-891.
36. Odibat Z, Momani S. A generalized differential transform method for linear partial Differential equations of fractional order. Applied Mathematics Letters. 2008; 21(2):194-199.
37. Odibat Z, Momani S, Erturk VS. Generalized differential transform method: application to differential equations of fractional order. Applied Mathematics and Computation. 2008; 197(2):467-477.
38. Podlubny I. Fractional differential equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. Academic Press. 1999.
39. Pourdarvish A. A reliable symbolic implementation of algorithm for calculating Adomian polynomials. Appl. Math. Comput. 2006; 172:545-550.
40. Soltanalizadeh B, Zarebnia M. Numerical analysis of the linear and nonlinear Kuramoto-Sivashinsky equation by using Differential Transformation method. Inter. J. Appl. Math. Mechanics. 2011; 7(12):63-72.
41. Song L, Zhang H. Application of homotopy analysis method to fractional KdV-Burgers-Kuramoto equation. Phys. Lett. A. 2007; 367(1-2):88-94.

42. Song L, Zhang H. Solving the fractional BBM-Burgers equation using the homotopy analysis method. *Chaos* Solitons Fractals. 2009; 40(4):1616-1622.
43. Tari A, Rahimi MY, Shahmoradb S, Talati F. Solving a class of two-dimensionallinear and nonlinear Volterra integral equations by the differential transform method. J. Comput. Appl. Math. 2009; 228:70-76.
44. Wazwaz AM. A reliable modification of Adomian decomposition method. Appl. Math. Comput. 1999; 102:77-86.
45. Wazwaz AM. A new algorithm for calculating Adomian polynomials for nonlinear Operators. Applied Mathematics and Computation. 2000; 111:33-51.
46. Wazwaz AM. The decomposition method for solving the diffusion equation subject to the classification of mass. IJAM. 2000; 3(1):25-34.
47. Wazwaz AM. Exact solutions to nonlinear diffusion equations obtained by the decomposition method. Appl. Math. Comput. 2001; 123:109-122.
48. Wazwaz AM. Partial Differential Equations :Methods and Applications. Saint Xavier University, Chicago, Illinois, USA. 2002.
49. Wazwaz AM. Adomian decomposition method for a reliable treatment of the Emden-Fowler equation. Appl. Math. Comput. 2005; 161:543-560.
50. Wazwaz AM. Adomian decomposition method for a reliable treatment of the Bratu-type equations. Appl. Math. Comput. 2005; 166:652-663.
51. Xu H, Cang J. Analysis of a time fractional wave-like equation with the homotopy analysis method. Phys. Lett. A. 2008; 372(8):1250-1255.
52. Xu H, Liao SJ, You XC. Analysis of nonlinear fractional partial differential equations with the homotopy analysis method. Commun. Nonlinear Sci. Numer. Simul. 2009; 14(4):1152-1156.
53. Zhou JK. Differential Transformation and Its Applications for Electrical Circuits. Huazhong University Press, Wuhan, China. 1986.
54. Zhu Y, Chang Q, Wu S. A new algorithm for calculating Adomian polynomials. Appl. Math. Comput. 2005; 169:402-416.