International Journal of Academic Research and Development
ISSN: 2455-4197
Impact Factor: RJIF 5.22
www.academicsjournal.com
Volume 2; Issue 5; September 2017; Page No. 643-646

Baire measures and its unique extension to a regular boral measure

Baljodh Singh
Assistant Professor, S.G.G.S Khalsa College, Mahilpur, Punjab, India

Abstract
A Baire measure is a probability measure on the Baire $\sigma - algebra$ over a normal Hausdorff space X. A Borel measure is a probability measure on the Borel $\sigma - algebra$ over a normal Hausdorff space X. In this paper we prove that every Baire measure has a unique extension to a regular Borel measure.

Keywords: baire measures, unique extension**,** borel measure

1. Introduction

Baire measure is a measure on $\sigma - algebr$ of Baire sets of a topological space X whose value on every compact Baire set is finite. In theory of measure and integration the Baire sets of a locally compact Hausdorff space form a $\sigma - algebra$ related to continuous functions on the space. There are in equivalent definitions of Baire sets which are coincide with case of locally compact and σ-compact Hausdorff spaces. If X be a topological space, Baire sets are those subsets of X belonging to the smallest $\sigma - algebra$ containing all zero sets in X where a zero set is defined as under:

Definition: A set Z $\subset X$ is called a zero set if Z = $f^{-1}_{(0)}$ for some continuous real valued function f on X.

Definition: Let X be a topological space then Baire $\sigma - algebra$ $\mathfrak{B}_0$(X) on X is the smallest $\sigma - algebra$ containing the pre-images of all continuous functions f: X$\rightarrow X$. And if there exist a measure μ on $\mathfrak{B}_0$(X) s.t. $\mu(X) < \infty$. Then μ is called a finite Baire measure on X. Further if $\mathfrak{B}(X)$ is the Borel $\sigma - algebra$ on X (i.e. the smallest $\sigma - algebra$ containing the open sets of X) then $\mathfrak{B}_0$(X)$\subset$ $\mathfrak{B}(X)$.

Definition: Borel sets are those sets of X belonging to the smallest $\sigma - algebra$ that contains all closed subsets of X. Clearly a Baire set is always a Borel set. But in many familiar spaces including all metric spaces the classes of all Baire sets and Borel sets are coincide.

Remark: If X be the metric space then $\mathfrak{B}_0$(X) = $\mathfrak{B}(X)$.

Regular Borel Measure: Let μ be a Borel measure on a space X and let $E \in \mathfrak{B}$.We say that the measure μ is outer regular on E if $\mu(E) = inf\{\mu(U): E \subset U, U \text{ is open}\}$ and we say that measure μ is inner regular on E if $\mu(E) = sup\{\mu(K) : K \subset U, K \text{ is compact}\}$. If μ is both inner and outer regular on E then we say that μ is regularon E. Further μ is called Regular Borel measure if it is regular on every Borel set. For example a Radon measure is a Borel measure which is

a. Finite on every compact set.
b. Outer regular on every Borel set.
c. Inner regular on every open set.

Preposition: Let μ be a Borel measure which is finite on compact sets. Then the following statements are equivalent.
1. μ is outer regular on σ-bounded sets.
2. μ is inner regular on σ-bounded sets.

$Proof: (1) \Rightarrow (2)$ Suppose that E is a bounded Borel set and E $\subset L$ Where L is compact. Assume that $\varepsilon > 0$. We have to prove that there is a compact set K $\subseteq E$ with μ (K) $\geq \mu$ (E)-ε. As the relative compliment L/E is bounded, by outer regularity there is an open set O$\supseteq L/E$ such that μ (O) $\leq \mu$ (L/E) +ε. It follows that K = L/O = L$\cap O^c$ is a compact set of E satisfying μ(K) = μ(L) -

μ(L$\cap O$) $\geq \mu$ (L)- μ(O) $\geq \mu$(L/E)-ε, as required.

In general let $E = E_1 \cup E_2 \cup E_3 \cup \ldots$ is a countable union of bounded Borel sets E_i. We may assume that the sets E_i are disjoint. If some of the E_i has finite measure, then by above we have Sup$\{\mu(K): K \subseteq E_i, K \in \mathcal{k}\} = \mu(E_i) = +\infty$, where $\mathcal{k}$ is the family of compact sets. Then Sup$\{\mu(K): K \subseteq E, K \in \mathcal{k}\} = \mu(E) = +\infty$ Proved.

But on the other hand if $\mu(E_i) < \infty$ for each i then for any $\varepsilon > 0$ we can find a sequence of compact sets $K_i \subseteq E_i$ with the property that $\mu(E_i) \leq \mu(K_i) + \frac{\varepsilon}{2^i}$.

Taking $L_n = K_1 \cup K_2 \cup \ldots\ldots \cup K_n$, it is clear that L_n is a compact subset of E for which $\mu(L_n) = \sum_{i=}^{n} \mu(K_i) \geq \sum_{i=}^{n} \mu(K_i) - \frac{\varepsilon}{2^i} \geq \sum_{i=}^{n} \mu(K_i) - \varepsilon$. Taking supremum over n we get S$up\ \mu(L_n) \geq \mu(E) - \varepsilon$. Which shows that μ is Inner regular on σ-bounded sets.

$(2) \Rightarrow (1)$ Let E be abounded Borel set. Then closure of E is $\bar{E}$ and is compact set and by single covring there exist a bounded open set U s.t.$\bar{E} \subseteq U$. Let $\varepsilon > 0$ then L/E is a bounded Borel set, then by Inner regularity there exist a bounded compact set K $\subseteq$ L/E *with* the property that $\mu(K) \geq \mu(L/E) - \varepsilon$. Let V = U/K = U$\cap K^c$ then V is a bounded and open which contains E and $\mu(V) = \mu(U \cap K^c) \leq \mu(L \cap K^c) = \mu(L) - \mu(K) \leq \left(\mu\left(\frac{L}{E}\right) - \varepsilon\right) = \mu(E) - \varepsilon$. As ε is arbitrary positive and this proves that μ is outer regular on bounded sets.

Further let E = $\cup_n E_n$ where each E_n is a bounded Borel set and each E_i is disjoint and $\mu(E_n) < \infty$ for all n. From the above we have a sequence of open sets $O_n \supseteq E_n$ such that $(O_n) \leq \mu(E_n) + \frac{\varepsilon}{2^n}$. Therefore the set E is contained in the union of O=$\cup_n O_n$ and we get $\mu(O) \leq \sum_{i=}^{n} \mu(O_i) \leq \sum_{i=}^{n} \mu(E_i) + \varepsilon = \mu(E) + \varepsilon$.Hence the proof.

Content: A real valued function defined on a $\sigma - algebra\ \mathcal{A}$ of sub sets of a space X is said to be a content on X if 1.) $\mu(A) > 0$ for all A $\in \mathcal{A}$.

2. $\mu(\emptyset) = 0$ and

3. $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$ for all $A_1, A_2 \in \mathcal{A}$.

Preposition: Let λ be content on X and μ be a regular Borel measure induced by λ, then the following statements are equivalent.

1. λ is regular content.
2. μ is an extension of λ.

Proof: Suppose that (1) holds i.e. λ is a regular content. Let C be any compact set, and $\varepsilon > 0$, By the regularity of λ, we can find a compact set D s.t.

C$\subset$D and $\lambda(D) \leq \lambda$(C) + ε .... (1)

Let λ^* be the outer measure induced by λ. Then let U = D then C$\subset$U $\subset$D and U is an open bounded Borel set, we have $\lambda^*(C) \leq \lambda^*(U)$ $\{Because\ \lambda^*\ is\ monotone\}$

$\leq \lambda(D)) \leq \lambda$(C) + ε $\{By\ (1)\}$

Hence $\lambda^*(C) \leq \lambda$(C) But $\mu(C) = \lambda^*(c)$ Therefore $\lambda^*(C) = \lambda$(C) i.e. $\mu(C) = \lambda$(C) which shows that μ is an extension of λ.

Now suppose that (2) holds, that means μ is an extension of λ. Hence λ is restriction of μ *and* μ is regular Borel measure. Hence by case (1) λ is a regular content.

Remark: Let C and D be two disjoint sets of X, and then there exist open bounded Baire sets U and V s.t. C $\subset$U and D $\subset$V.

Proof: Let $x \in C$ then $x \notin D$ we can find disjoint open sets U_x and V_x s.t.. $x \in U_x$, D$\subset V_x$. It is clear that $\{V_x / x \in C\}$ is an open cover for C and C is compact, hence there exist a finite sub cover x_1, x_2......... x_n s.t. C $\subset \cup_{i=1}^{n} U_{x_i}$. Let $U^* = \cup_{i=1}^{n} U_{x_i}$ *and* $V^* = \cap_{i=1}^{n} V_{x_i}$ then U^* *and* V^* are disjoint open sets and C $\subset U^*$ and D $\subset V^*$, By Baire sandwich theorem there exist open bounded Baire sets U and V s.t. C $\subset$U $\subset U^*$ and D $\subset$V D $\subset V^*$ and obviously U and V are disjoint.

Main Result: Every Baire measure has a unique extension to a regular Borel measure.

Lemma: Let ν be any Baire measure on X. Define for compact set C $\lambda(C) = Inf\{\nu(C)/ C \subset U, U \text{ is open Baire set}\}$. Then λ is a regular content and $\lambda(D) = \mu(D)$ for every compact G_δ set D.

Proof of the Lemma: (1) Since $\nu \geq 0$ then obviously $\lambda \geq 0$.

Let C be any compact set, By Baire sandwich theorem we can find an open Baire set U and a compact G_δ set D s.t.C $\subset$U $\subset D$. Which gives that $\lambda(C) \leq \nu(U) \leq \nu(D) < \infty$ which shows that λ is real valued.

(2) Suppose C and D are two compact sets and C$\subset D$. Let U be any open Borel set s.t. D $\subset$U then C $\subset U \Rightarrow \lambda(C) \leq \nu(U) \Rightarrow \lambda(C) = Inf\{\nu(U)/ U \text{ is open Baire set and } D \subset U\}$
.
$\Rightarrow \lambda(C) \leq \lambda(D) \Rightarrow \lambda$ is monotone.

(3) Let C and D be compact sets and U be any open Baire set s.t. C $\subset U$ and V any open Baire set s.t. D $\subset V$ then C$\cup$D $\subset$ U$\cup$V
$\Rightarrow \lambda(C \cup D) \leq \nu(U \cup V) \leq \nu(U) + \nu(V)$
$\Rightarrow \lambda(C \cup D) \leq \inf\{\nu(U)\} + \inf\{\nu(V)\} \Rightarrow \lambda(C \cup D) \leq \lambda(C) + \lambda(D) \Rightarrow \lambda$ is sub additive.

(4) Let C and D be any two disjoint compact sets then by above remark we can find disjoint open bounded Baire sets U and V s.t. C $\subset$U and D $\subset$V. Let W be an open Baire set s.t. C$\cup$D $\subset$W then C $\subset$ U$\cap W$ and D $\subset$ V$\cap W$.
Since $W \supset (U \cap W) \cup (V \cap W)$, we get $\nu(W) \geq \nu[(U \cap W) \cup (V \cap W)] = \nu(U \cap W) + \nu(V \cap W) \geq \lambda(C) + \lambda(D)$.
$\Rightarrow \text{Inf}\{\nu(W)\} \geq \lambda(C) + \lambda(D)$ then from (3) we get $(C \cup D) \leq \lambda(C) + \lambda(D)$. Hence we get $(C \cup D) = \lambda(C) + \lambda(D)$. This proves that λ is content.

(5)**Regularity**: Let C be any compact set, $\varepsilon > 0$ then by definition of λ we can find an open Baire set U such that C$\subset U$ and$(U) \leq \lambda(C) + \varepsilon$. By Baire sandwich theorem we can find an open Baire set V and a compact G_δ set D s.t. C$\subset V \subset D \subset U$. Then C$\preccurlyeq D$ and $\lambda(D) \leq \nu(U) \leq \lambda(C) + \varepsilon$. $\Rightarrow \lambda(C) = Inf\{\lambda(D)/C \preccurlyeq D, D \text{ is compact}\}$. This proves that λ is regular content.

(6)**Finally:** Let D be any compact G_δ set. Let (U_n) be any sequence of open sets s.t. D = $\cap_{i=1}^{n}(U_n)$ for each n, D $\subset U_n$, By Baire sandwich theorem we can find an open Baire set V_n and compact G_δ set D_n s.t. D $\subset V_n \subset D_n \subset U_n \Rightarrow$ D=$\cap_{i=1}^{n}(V_n)$.
Define $W_n = \cap_{i=1}^{n}(V_i)$. Then (W_n) is a monotone decreasing sequence of open bounded Baire sets s.t. $(W_n) \rightarrow$ D
$\Rightarrow \nu(W_n)) \rightarrow \nu$(D). as D$\subset W_n$ for all n $\Rightarrow \lambda(D) \leq \lim_{n\to\infty} \nu(W_n) \Rightarrow \lambda(D) \leq \nu(D)$ .... (*)
If V be any Open Baire set s.t. D$\subset$ V then $\nu(D) \leq \nu(V) \Rightarrow \nu(D) \leq \text{Inf}\{\nu(W)\} \Rightarrow \nu(D) \leq \lambda(D)$ .... (**)
From (*) and (**) we get $\lambda(D) = \nu(D)$.

Proof of the Main Theorem: Let ν be any Baire measure on X.
Define for compact set C, $\lambda(C) = Inf\{\nu(U)/C \subset U, U \text{ is Baire set}\}$. Then λ is a regular content. Let μ be the regular Borel measure induced by λ. Then Let μ is an extension of λ.Let ν' be the Baire restriction of μ. Let D be any compact G_δ set,
then $\nu(D) = \lambda(D) = \nu'(D)$ [By above lemma]
$\Rightarrow \nu(E) = \nu'(E)$ For all Baire sets E. $\Rightarrow \nu(E) = \mu(E)$ For every Baire set E.
i.e. μ is an extension of ν. Thus Baire measure ν has been extended to a regular Borel measure μ.

Uniqueness: Let μ_1 and μ_2 be two regular Borel measures such that $\mu_1(D) = \mu_2$(D) for every compact G_δ set D.
To prove above we have to show that $\mu_1(E) = \mu_2$(E) for every Borel set E. For this it is suffices to prove that $\mu_1(C) = \mu_2$(C) for every compact set C.

Let C be any compact set. Since $\mu_1$is regular Borel measure we can find a compact G_δ set D_1 such that C $\subset D_1$ and $\mu_1(C) = \mu_1(D_1)$ .... (1)

By the same argument we can find a compact G_δ set D_2 such that C $\subset D_2$ and $\mu_2(C) = \mu_2(D_2)$ .... (2)

Define D=$D_1 \cap D_2$, then D is a compact G_δ set and C $\subset D_1$ and C $\subset D_2 \Rightarrow C \subset D$ shows that $\mu_1(C) = \mu_1(D_1) \leq \mu_1(D) = \mu_2(D) \leq \mu_2(D_2) = \mu_2(C) \Rightarrow \mu_1(C) \leq \mu_2(C)$

By the same argument $\mu_2(C) \leq \mu_1(C)$.

Hence proved that $\mu_1(C) = \mu_2(C)$. Hence the theorem.

References

1. Choquet G. Lectures on Analysis Volume II: Representation Theory, of Mathematics, Lecture Note Series. W.A. Benjamin, Inc, London. 1969, 2.
2. Ghasemi M, Kuhlmann S, Marshall M. Application of Jacobi's representation theorem to locally multiplicatively convex topological real algebras. J Funct. Anal. 2014; 266(2):1041-1049.
3. William Arveson. Notes on measure and integration in locally compact spaces. University of California, USA, 1996.
4. Halmos PR. *Measure Theory*, Van Nostr and, Princeton, 1950.
5. Royden HL. *Real Analysis*, third edition, Macmillan, New York, 1988.
6. Mattila P. Geometry of sets and measures in euclidean spaces". Cambridge Studies in Advanced Mathematics, 44. Cambridge University Press, Cambridge, 1995.