



## On a conjecture on linear systems

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### Abstract

In a remark to Green’s conjecture, Paranjape and Ramanan analysed the vector bundle  $E$  which is the pullback by the canonical map of the universal quotient bundle  $T_{P^{g-1}}(-1)$  on  $P^{g-1}$  and stated a more general conjecture and proved it for the curves with Clifford Index 1 (trigonal and plane quintics). In this paper, we state the conjecture for general linear systems and obtain results for the case of hyper-elliptic curves.

**Keywords:** green’s conjecture; linear systems; hyper-elliptic curves

### 1. Introduction

Let  $C$  be a smooth projective curve of genus  $g \geq 2$  over a field  $k$  and let  $K$  be the canonical line bundle on  $C$ . In [4], Green made a conjecture which relates two aspects: Koszul cohomology (an algebraic aspect) and Clifford Index  $\gamma_C$  (a geometric aspect) of a curve. This conjecture [4] is equivalent to the following [6]: Let  $E_K$  be the pullback by

$i$  the canonical map  $C \rightarrow P^{g-1}$  of universal quotient bundle on  $P^{g-1}$ . Then the map

$\Lambda^i(C, E_K) \rightarrow (C, \Lambda^i E_K)$  is surjective  $\forall i \leq \gamma_C$ . Paranjape and Ramanan [6] studied the vector bundle  $E_K$  (stability properties). They also proved that all sections of  $\Lambda^i E_K$  which are locally decomposable are in the image of  $\Lambda^i(E_K) \forall i \leq \gamma_C$ . Let  $i,K$  be the cone of locally decomposable sections of  $\Lambda^i E_K$ . In [5], Hulek *et al.* stated a conjecture.

*Conjecture 1.1.*  $i,K$  spans  $(\Lambda^i E_K) \forall i$  and for all curves.

This is stronger than Green’s conjecture. They proved it for curves with Clifford index 1 (trigonal curves and plane quintics). Conjecture 1.1 is trivial in case of hyperelliptic curves, since  $E_K$  is the  $(g - 1)$ -fold direct sum of the hyper-elliptic line bundle. Vector bundle  $E_K$  is semi-stable (even stable if  $C$  is not hyper-elliptic). In a remark to conjecture made in [5], Eusen and Schreyer [3] asked a more general question, whether  $(\Lambda^i N)$  is spanned by locally decomposable sections and holds for every (stable) globally generated vector bundle  $N$  on every curve  $C$ . They gave counter examples to this more general question [3]. By broadening our view point, in this paper we state a conjecture for general linear systems. Let  $C$  be a smooth curve of genus  $g \geq 2$  and let  $L$  be a globally generated line bundle on  $C$ . The evaluation map gives rise to an exact sequence

$0 \rightarrow E^* \rightarrow (L)_C \rightarrow L \rightarrow 0$   
 where  $E^*$  is locally free of rank  $h^0(L) - 1$ . Let  $i$  be the cone of locally decomposable sections of  $\Lambda^i E$ . We state as follows: –  $i$

*Conjecture 1.2.*  $i$  spans  $(\Lambda^i E) \forall i$  and for all curves.

In this paper, we prove Conjecture 1.2 in case of hyperelliptic curves for the line bundles with degree large enough and  $i \leq g$ .

**Theorem 1.** *Let  $C$  be a smooth hyper-elliptic curve of genus  $g \geq 2$  and let  $L$  be a globally generated line bundle on  $C$  of degree  $d \geq 2g + 1$  such that  $H^1(L \otimes T^{-2}) = 0$ , where  $T$  is the hyper-elliptic line bundle on  $C$ . The evaluation map gives rise to an exact sequence*

$0 \rightarrow E^* \rightarrow (L)_C \rightarrow L \rightarrow 0, \tag{1}$   
 where  $E^*$  is locally free of rank  $h^0(L) - 1$ . Let  $i$  be the cone of locally decomposable sections of  $\Lambda^i E$ . Then  $i$  spans  $(\Lambda^i E) \forall i \leq g$ .

## 2. Geometry of the hyper-elliptic curve

Since  $C$  is hyper-elliptic of genus  $g \geq 2$ , thus  $g_2^1$  on  $C$  is unique. Let  $\pi : C \rightarrow \mathbb{P}^1$  be the associated 2-sheeted covering,  $T_{\mathbb{P}^1}(1)$  is the unique  $g^1_2$ .

Consider the rank 2 vector bundle  $W$  on  $\mathbb{P}^1$ , where  $W := \pi_*L$ . Since

$$\chi(L) = \chi(\pi_*L),$$

so

$$d + 1 - g = \text{rk}(W) \frac{\deg W}{\text{rk}W} + 1 - g_{\mathbb{P}^1} = 2 \frac{\deg W}{2} + 1$$

which gives

$$\deg W = d - g - 1.$$

Thus,

$$\begin{aligned} \det W &\sim_{\mathbb{P}^1} \mathcal{O}_{\mathbb{P}^1}(d - g - 1), \\ W &\sim W^*(d - g - 1). \end{aligned}$$

$$= \dots \leq 2$$

Since  $\deg W = d - g - 1$ , there is a unique integer  $x = \lfloor \frac{d-g-1}{2} \rfloor$  such that

$$W \sim_{\mathbb{P}^1} \mathcal{O}_{\mathbb{P}^1}(x) \otimes \mathcal{O}_{\mathbb{P}^1}(d - g - 1 - x). \tag{2}$$

**Remark 1.**

(i)  $x$  is the least integer such that

$$H^1(W(-2-x)) = H^1(\pi_*L(-2-x)) = H^1(L \otimes T^{-2-x}) = 0.$$

In particular, this implies that  $\deg(L \otimes T^{-2-x}) \leq 2g - 2$  and thus we have

$$\frac{d - 2g - 2}{2} \leq x \leq \frac{d - g - 1}{2}.$$

(ii) Since  $H^1(L \otimes T^{-2}) = 0$ , thus  $x > 0$ , so we have

$$\max\{1, \frac{d - 2g - 2}{2}\} \leq x \leq \frac{d - g - 1}{2}.$$

which implies that both  $W$  and  $W(-1)$  are globally generated.

(iii) Also,  $H^1(L \otimes T^{-2}) = 0$  implies  $H^1(L) = 0$ , thus by Riemann–Roch theorem, we have

$$h^0(L) = d - g + 1 \tag{3}$$

and  $\text{rank}(E) = h^0(L) - 1 = d - g \geq 3$  (since  $d \geq 2g + 1$  and  $g \geq 2$ ).

(iv) We have

$$\begin{aligned} (W(-1)) &\cong (\pi^*(L \otimes T^{-1})) \\ &\cong \pi^*(L) \otimes \pi^*(T^{-1}) \\ &\cong \pi^*(L) \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \\ &\cong \pi^*(L \otimes T^{-1}) \end{aligned}$$

Also  $H^1(L \otimes T^{-2}) = 0$  gives  $H^1(L \otimes T^{-1}) = 0$ . Thus, by Riemann–Roch theorem, we have

$$h^0(L \otimes T^{-1}) = d - g - 1$$

i.e., we have

$$h^0(W(-1)) = d - g - 1. \tag{4}$$

Since  $W(-1)$  is globally generated and  $(W) \cong (L)$ , we have a surjection  $(L)_{\mathbb{P}^1} \rightarrow W \rightarrow 0$ ,

which is an isomorphism for sections. Since  $W(-1)$  is a globally generated bundle on  $\mathbb{P}^1$ ,  $W$  is very ample, i.e., we get an inclusion

$$\begin{aligned} \mathbb{P}(W^*) &\subset \mathbb{P}(L^*) \subset \mathbb{P} \\ &\rightarrow \mathbb{P} \end{aligned} \tag{5}$$

Also we have a surjection

$$\pi^*W \rightarrow L \rightarrow 0.$$

In other words, we have a subbundle of  $\pi^*(W^*)$  that is isomorphic to  $L^{-1}$ . This gives a morphism from  $C$  to  $\mathbb{P}(W^*)$  with the property that the pullback of  $\mathcal{O}_W(1)$  to  $C$  is  $L$ . Also the composite of this morphism with the projection  $p : \mathbb{P}(W^*) \rightarrow \mathbb{P}^1$  is  $\pi$ . Since the induced map

$$\begin{aligned} (\mathbb{P}(W^*), \mathcal{O}_W(1)) &\cong (\mathbb{P}^1, W) \cong (C, L) \\ \mathcal{O} &\rightarrow \end{aligned}$$

is an isomorphism,  $C$  is actually embedded in  $\mathbb{P}(W^*)$ . Let us denote the image of  $\mathbb{P}(W^*)$  in  $\mathbb{P}$  by  $S$ . We return to the ruled surface  $p : \mathbb{P}(W^*) \rightarrow \mathbb{P}^1$ . By (5), there is an embedding  $\mathbb{P}(W^*) \subset \mathbb{P} = \mathbb{P}(L^*)$  with hyperplane section  $\tau = \mathcal{O}_W(1)$ . Note that  $\tau^2 = \deg W = d - g - 1$ . Let  $h$  be the class of a fibre of the projection  $p$ . Then the Picard group of  $\mathbb{P}(W^*)$  is generated by  $\tau$  and  $h$ . Since  $C$  is a secant (2-section) of  $\mathbb{P}(W^*)$ , its class is of the form  $\mathcal{O}_W(2) \otimes p^*\mathcal{O}_{\mathbb{P}^1}(m)$ . To compute  $m$ , we note that  $d = C \cdot \tau = 2\tau^2 + m$ . Thus  $m = 2g - d + 2$  and we have the following proposition.

**PROPOSITION 2.1**

*There are inclusions*

$$C \subset S \subset \mathbb{P}$$

*With the following properties:*

1. *The restriction of  $\mathcal{O}_{\mathbb{P}}(1)$  to  $S$  is  $\mathcal{O}_W(1)$ ;*
2. *The restriction of  $\mathcal{O}_{\mathbb{P}}(1)$  to  $C$  is  $L$ ;*
3. *Both restrictions induce isomorphisms of the corresponding linear systems;*
4. *The divisor class on  $S$  defined by  $C$  is  $\mathcal{O}_W(2) \otimes p^*\mathcal{O}_{\mathbb{P}^1}(-d + 2g + 2)$ .*

*Notation.* We will use the notation  $U$  for the vector space  $(L \otimes T^{-1})$ , i.e. we have

$$(W(-1))_{\mathbb{P}^1} = U. \tag{6}$$

and by (4), we have

$$\dim U = d - g - 1. \tag{7}$$

**3. Computation of dimensions**

In order to prove the conjecture, we want to relate the sections of  $\Lambda^i E$  to the sections of a suitable vector bundle on  $P^1$ .

*Lemma 3.1. Let  $F$  be a vector bundle on  $P^1$  that is globally generated. Then the evaluation sequence is*

$$0 \rightarrow (F(-1)) \otimes_{O_{P^1}}(-1) \rightarrow (F)_{P^1} \rightarrow F \rightarrow 0.$$

*Proof.*  $F$  is a sum of line bundles of degree  $\geq 0$ . Thus it remains to check for line bundles, which is easy.

We want to apply this lemma to  $W$ .

$$0 \rightarrow (W(-1)) \otimes_{O_{P^1}}(-1) \rightarrow (W)_{P^1} \rightarrow W \rightarrow 0. \tag{8}$$

Pulling back the evaluation sequence for  $W$  on  $P^1$  to  $C$  and using (6) and the fact that

$$(\pi^*L) \cong \pi^*(L), \text{ we get}$$

$$0 \rightarrow U \otimes T^{-1} \rightarrow (L)_C \rightarrow \pi^*W \rightarrow 0. \tag{9}$$

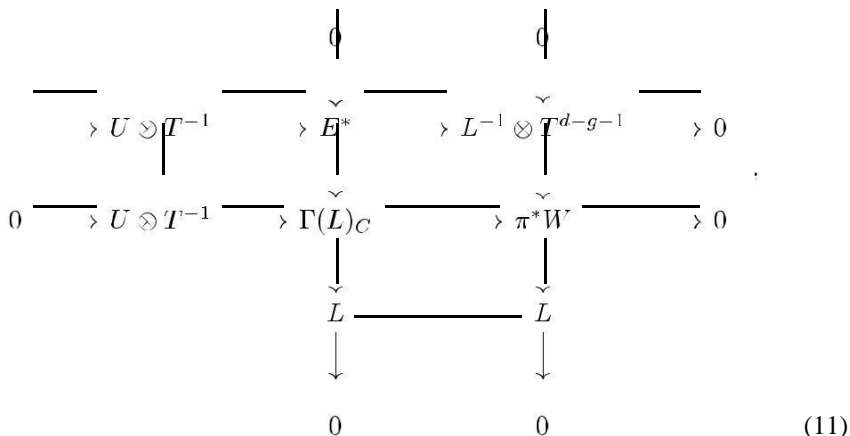
Also, we have a surjective map  $\pi^*W \rightarrow L \rightarrow 0$ . Let  $Y$  be the kernel of  $\pi^*W \rightarrow L \rightarrow 0$ , i.e. we have

$$\begin{aligned} 0 \rightarrow Y \rightarrow \pi^*W \rightarrow L \rightarrow 0, \\ \wedge^2(\pi^*W) \cong \wedge^2 Y \oplus Y \otimes L, \\ \pi^*(\wedge^2 W) \cong \wedge^2 Y \oplus Y \otimes L, \\ \pi^*(\wedge^2(d-g-1)) \cong \wedge^2 Y \oplus Y \otimes L, \\ \pi^*(T^{d-g-1}) \cong Y \oplus L^{-1} \otimes T^{d-g-1}. \end{aligned}$$

Thus, we have

$$0 \rightarrow L^{-1} \otimes T^{d-g-1} \rightarrow \pi^*W \rightarrow L \rightarrow 0 \tag{10}$$

and we get a following commutative diagram



where the left vertical map is the evaluation map.

Dualizing the diagram (11), we get

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
 0 & \longrightarrow & L \otimes T^{-(d-g-1)} & \longrightarrow & E & \longrightarrow & U^* \otimes T \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \parallel \\
 & & & & & & \\
 0 & \longrightarrow & \pi^* W^* & \longrightarrow & \Gamma(L)_C^* & \longrightarrow & U^* \otimes T \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \\
 & & L^{-1} & \longleftarrow & L^{-1} & & \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & \\
 & & & & & & (12)
 \end{array}$$

The first line of (12) gives rise to an exact sequence

$$0 \rightarrow L \otimes T^{-(d-g-1)} \otimes \Lambda^{i-1} U^* \otimes T^{i-1} \rightarrow \Lambda^i E \rightarrow \Lambda^i U^* \otimes T^i \rightarrow 0. \quad (13)$$

Since,  $\pi^* W^*$  is a rank 2 bundle we only get a filtration consisting of the following two exact sequences:

$$0 \rightarrow \Lambda^2 \pi^* W^* \otimes \Lambda^{i-2} U^* \otimes T^{i-2} \rightarrow \Lambda^i (L)_C^* \rightarrow L_i \rightarrow 0, \quad (14)$$

$$0 \rightarrow \pi^* W^* \otimes \Lambda^{i-1} U^* \otimes T^{i-1} \rightarrow L_i \rightarrow \Lambda^i U^* \otimes T^i \rightarrow 0. \quad (15)$$

Since the second horizontal sequence of (12) is the pullback via  $\pi$  of the dual of the sequence (8), both the above sequences come from  $P^1$ , i.e. there exists a vector bundle  $L_i$  on  $P^1$ , such that  $L_i = \pi^* L_i$  and the sequences

$$\begin{array}{ccccccc}
 0 & & 2W^* & & i-2U^* & & (i-2) & & i & & (L)^* & & 1 & & L & & 0, & (16) \\
 \rightarrow & \Lambda & \otimes \Lambda & \otimes O & \rightarrow & \Lambda P & \rightarrow & \Lambda P & \rightarrow & i & \rightarrow & & & & & & & 
 \end{array}$$

$$0 \rightarrow W^* \otimes \Lambda^{i-1} U^* \otimes O(i-1) \rightarrow L_i \rightarrow \Lambda^i U^* \otimes O(i) \rightarrow 0 \quad (17)$$

are such that (14) and (15) are pullbacks of (16) and (17) respectively. Dualizing (1), we have

$$0 \rightarrow L^{-1} \rightarrow (L)^*_C \rightarrow E \rightarrow 0.$$

Thus the map  $\Lambda^i (L)^*_C \rightarrow \Lambda^i E$  is surjective. Also we have  $\Lambda^i (L)^*_C \rightarrow L_i \rightarrow 0$ . The maps  $\Lambda^i (L)^*_C \rightarrow \Lambda^i E \rightarrow 0$  factors through  $L_i = \pi^* L_i$ . Thus we get the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
 0 \rightarrow & L \otimes T^{-(d-g-1)} \otimes \Lambda^{i-1} U^* \otimes T^{i-1} & \longrightarrow & \Lambda^i E & \longrightarrow & \Lambda^i U^* \otimes T^i \rightarrow 0 & \\
 & \uparrow & & \uparrow & & \parallel & \\
 0 \longrightarrow & \pi^* W^* \otimes \Lambda^{i-1} U^* \otimes T^{i-1} & \longrightarrow & \pi^* L'_i & \longrightarrow & \Lambda^i U^* \otimes T^i \rightarrow 0 & \\
 & \uparrow & & \uparrow & & & \\
 & L^{-1} \otimes \Lambda^{i-1} U^* \otimes T^{i-1} & \longleftarrow & L^{-1} \otimes \Lambda^{i-1} U^* \otimes T^{i-1} & & & \\
 & \uparrow & & \uparrow & & & \\
 & 0 & & 0 & & & \\
 & & & & & & (18)
 \end{array}$$

where the top horizontal sequence is (13), middle horizontal sequence is (15), left vertical sequence is obtained by dualizing (10) and tensoring it with  $\Lambda^{i-1} U^* \otimes T^{i-1}$ . Now let us compute the dimensions of the spaces  $(L_i)$  and  $(\Lambda^i E)$  for  $i \leq d - g (= \text{rank } E)$ .

Lemma 3.2. When  $d \geq 2g + 1$ , we have

$$\dim(L)_i = \binom{d-g+1}{i} + \binom{d-g-1}{i-2} - \binom{d-i-g}{i}$$

for  $i \leq d - g$

Proof. Consider (16). Then

$$0 \rightarrow \Lambda^{i-2} U^* \otimes O(i-2) \rightarrow \Lambda^{i-1} U^* \otimes O(i-1) \rightarrow \Lambda^i U^* \otimes O(i) \rightarrow 0$$

Since

$$\det W \sim \mathcal{O}_{\mathbb{P}^1}(d-g-1),$$

we get

$$\det W^* \sim \mathcal{O}_{\mathbb{P}^1}(-(d-g-1)).$$

Hence we get

$$0 \rightarrow \Lambda^{i-2} U^* \otimes O(i-d+g-1) \rightarrow \Lambda^i U^* \otimes O(i) \rightarrow L_i \rightarrow 0.$$

Since

$$i \leq d - g, h^0(O(i-d+g-1)) = 0,$$

therefore,

$$h^0(L_i) = h^0(\Lambda^i U^* \otimes O(i)) + h^1(\Lambda^{i-2} U^* \otimes O(i-d+g-1)).$$

By (7), we have

and by (3)

$$\dim U = d - g - 1,$$

$$h^0(L) = d - g + 1,$$

$$h^1(O(i-d+g-1)) = d - i - g.$$

Thus

$$\dim(L)_i = \binom{d-g+1}{i} + \binom{d-g-1}{i-2} - \binom{d-i-g}{i}$$

### 3.1 Syzygies of the curve

The syzygies of canonically embedded curves were computed by Schreyer [8]. Based on the parallel idea, we compute the syzygies of the curve  $C$ . For this, let

$$R = \bigoplus_{i=1}^{\infty} (C, L^i)$$

be the homogeneous coordinate ring of  $C$  with respect to  $L$  and

$$S = \text{Sym}_{n \geq 0} (C, L) = \mathcal{O}_{\mathbb{P}^1}(n).$$

Let

$$0 \rightarrow F_t \rightarrow \dots \rightarrow F_0 \rightarrow R \rightarrow 0 \tag{19}$$

be a minimal free resolution of the graded  $S$ -module  $R$ . Then  $F_i = \sum_j S(-j)^{\beta_{ij}} =$

$\sum_j M_{ij} \otimes S(-j)$ , where  $M_{ij}$  is a  $k$ -vector space of  $\dim \beta_{ij}$  and  $S(-j)$  is the free

$S$ -module with one generator in degree  $j$ . The resolution (19) is equivalent to the free resolution of  $O_C$  as an  $O_{P^d-g}$ -module:

$$0 \rightarrow \sum_j O(-j)^{\beta_{0j}} \rightarrow \dots \rightarrow \sum_j O(-j)^{\beta_{0j}} \rightarrow O_C \rightarrow 0.$$

To find this resolution, one starts with the exact sequence

$$0 \rightarrow O_S(-2\tau + (d - 2g - 2)h) \rightarrow O_S \rightarrow O_C \rightarrow 0$$

(see Proposition 2.1). The idea is to first resolve the sheaves  $O_S$  and  $O_S(-2\tau + (d - 2g - 2)h)$  as  $O_{P^d-g}$  modules and then form a mapping cone. The result turns out to be a minimal resolution of  $O_C$ . Firstly, we will recall from [2] the description of the syzygies of these sheaves.

Let  $\zeta = O(e_1) \otimes O(e_2) \otimes \dots \otimes O(e_s)$  be a locally free sheaf of rank  $s$  on  $P^1$ , and let  $p_\zeta : P(\zeta) \rightarrow P^1$  denote the corresponding  $P^{s-1}$  bundle. A rational normal scroll  $X$  of type  $S(e_1, e_2, \dots, e_s)$  with  $e_1 \geq e_2 \geq \dots \geq e_s \geq 0$  and

$$f = e_1 + e_2 + \dots + e_s \geq 2$$

is the image of  $P(\zeta)$  in  $P^r = P(H^0(P(\zeta), O_{P(\zeta)}(1)))$ :

$$j : P(\zeta) \rightarrow X \subset P^r, \quad r = f + s - 1.$$

The Picard group of  $P(\zeta)$  is generated by the hyper-plane class  $H = [j^* O_{P^r}(1)]$  and the ruling  $R = [p_\zeta^* O_{P^1}(1)]$ :

$$\text{Pic } P(\zeta) = \mathbb{Z}H \oplus \mathbb{Z}R.$$

The intersection product is given by

$$H^s = f, \quad H^{s-1} \cdot R = 1, \quad R^2 = 0.$$

We recall from [2], the description of the syzygies of the sheaves

$$O_X(aH + bR) := j_* O_{P(\zeta)}(aH + bR), \quad a, b \in \mathbb{Z} \text{ regarded as } O_{P^r} \text{-modules, at least in case } b \geq -1.$$

Let  $\mathcal{F} : F \rightarrow G$

be a map of locally free sheaves of rank  $f$  and  $g, f \geq g$ , respectively on a smooth variety  $V$ . We recall from [1] the family of complexes  $\zeta^b, b \geq -1$  of locally free sheaves on  $V$ , which resolve the  $b$ -th symmetric power of coker under suitable hypothesis on  $\mathcal{F}$ .

Define the  $j$ -th term in the complex  $\zeta^b$  by

$$\zeta^b_j = \mathcal{N} F \otimes S_{b-j} G, \quad \text{for } 0 \leq j \leq b,$$

$$\zeta^b_j = \mathcal{N}^{+g-1} F \otimes D_{j-b-1} G^* \otimes \Lambda^g G^*, \quad \text{for } j \geq b + 1$$

and differential

$$\zeta^b_j \rightarrow \zeta^b_{j-1}$$

by multiplication with  $\in H^0(V, F^* \otimes G)$  for  $j = b + 1$  and  $\wedge^g \in H^0(V, \wedge^g F^* \otimes \wedge^g G)$  for  $j = b + 1$  in the appropriate term of the exterior ( $\wedge F$ ), symmetric ( $SG$ ) or divided power ( $DG$ ) algebra.

PROPOSITION 3.3 [2]

$\zeta^b(a)$  for  $b \geq -1$  is the minimal resolution of  $O_X(aH + bR)$  as an  $O_{Pr}$ -module, where  $\zeta^b(a) = \zeta^b \otimes O_{Pr}(a)$ .

3.2 Minimal resolution of  $O_C$

We have

$$C \subset S \subset P = P(C, L^*),$$

$C$  is contained in a 2-dimensional rational normal scroll  $S$  of type  $S(e_1, e_2)$  and degree  $f = e_1 + e_2 = d - g - 1 \geq 2$ .  $C$  is a divisor of class

$$C \sim 2H - (f - (g + 1))R \quad \text{on } S.$$

The mapping cone [8]

$$\zeta^{f-(g+1)}(-2) \rightarrow \zeta^0$$

is the minimal resolution of  $O_C$  as an  $O_{Pd-g}$ -module. We consider

$$: F \otimes O_{Pd-g}(-1) \rightarrow G \otimes O_{Pd-g}$$

be the map of locally free sheaves, where  $F$  is a vector space of dimension  $f = d - g - 1$  and  $G$  is a vector space of dimension 2.

Firstly, we will compute

$$\zeta^{f-(g+1)}(-2) = \zeta^{d-2g-2} \otimes O(-2).$$

Now,

$$N^j(F \otimes O(-1)) \otimes S_{d-2g-2-j}(G \otimes O), \quad 0 \leq j \leq d - 2g - 2,$$

$$\zeta_j^{d-2g-2} = N^{j+1}(F \otimes O(-1)) \otimes D_{j-d+2g+2-1}(G \otimes O)^* \otimes \wedge^2(G \otimes O)^*,$$

$$j \geq d - 2g - 1.$$

Since  $j + 1$  can be at most  $d - g - 1$ , we have

$$N^j(F \otimes O(-1)) \otimes S_{d-2g-2-j}(G \otimes O), \quad 0 \leq j \leq d - 2g - 2,$$

$$\zeta_j^{d-2g-2} = \wedge^{j+1}(F \otimes O(-1)) \otimes D_{j-d+2g+2-1}(G \otimes O)^* \otimes \wedge^2(G \otimes O)^*,$$

$$d - 2g - 1 \leq j \leq d - g - 2.$$

Similarly we can compute  $\zeta^0$ ,

$$\zeta^0 = N^j(F \otimes O(-1)) \otimes S_{0-j}(G \otimes O), \quad j = 0,$$

$$\zeta^j = \wedge^{j+1}(F \otimes O(-1)) \otimes D_{j-1}(G \otimes O)^* \otimes \wedge^2(G \otimes O)^*, \quad 1 \leq j \leq d - g - 2.$$



Now,

$$\begin{aligned} H^1(\wedge^i E) &= H^1(\wedge^{d-g-i} E^* \otimes L) \quad (\text{rank } E = d - g) \\ &= H^0(\wedge^{d-g-i} E \otimes L^{-1} \otimes K)^*. \end{aligned}$$

Thus,

$$h^1(\wedge^i E) = h^0(\wedge^{d-g-i} E \otimes L^{-1} \otimes K).$$

Since

$$M_{p,p^*+2} \approx H^0(\wedge^{p+1} E \otimes L^{-1} \otimes K),$$

we have

$$M^* = H^0(\wedge^{d-g-i} E \otimes L^{-1} \otimes K)$$

$$\wedge^{d-g-i-1, d-g-i+1} = \wedge \otimes \otimes$$

and

$$\dim M^* = h^0(\wedge^{d-g-i} E \otimes L^{-1} \otimes K) = h^1(\wedge^i E).$$

In order to compute  $h^1(\wedge^i E)$ ,

i.e. look at degree

$$\dim_{g-i} \zeta^0_{d-g-i-1} =$$

$\wedge$  component of the term  $(\zeta^{d-2g-2}(-2))$

$\zeta^{d-g-i-2}$  in the minimal free resolution of

For  $i \leq g-1$ ,  $d-g-i-2 \geq d-2g-1$ , we have

$$\begin{aligned} \dim M^* &= \dim \wedge^{d-g-i-1, d-g-i+1} (F(-1)) \otimes O(-g-i-1) \\ &\quad \times (G \otimes O)^* \otimes \wedge^2(G \otimes O)^* \otimes O(-2) \\ &= \binom{d-g-1}{i} \binom{g-i}{-} \end{aligned}$$

since the contribution of term  $\zeta^0_{d-g-i-1}$  in degree  $(d-g-i+1)$  in the minimal free resolution of  $O_C$  is zero.

For  $i \geq g$ ,

$$\dim M^* = 0, 11$$

since the contribution of both terms  $\zeta^0_{d-g-i-1}$  and  $\zeta^0_{d-g-i-2}(-2)$  in degree  $(d-g-i+1)$

in the minimal free resolution of  $O_C$  is zero. Thus for  $i < g$ ,

$$\begin{aligned}
 h^0(\Lambda^i E) &= \binom{d-g-1}{i} \binom{g-d-2i-1}{-} + \binom{d-g-1}{i} \binom{1-2i-g}{-} \\
 &+ h^1(\Lambda^i E) \\
 &= \binom{d-g-1}{i} \binom{g-d-2i-1}{-} + \binom{d-g-1}{i} \binom{1-i}{+} \\
 &- \binom{d-g-1}{i} \binom{g-i}{-} + \binom{d-g-1}{i} \binom{g-i}{-} \\
 &= \binom{d-g-1}{i-1} \binom{g-d+2i+1}{+} + \binom{d-g-1}{(1+i)i} \\
 &+ \binom{d-g+1}{i} \binom{d-g+1}{-} \\
 &= \binom{d-g+1}{i} + \binom{d-g-1}{i} \binom{d-i-g}{2}
 \end{aligned}$$

and for  $i \geq g$ ,

$$h^0(\Lambda^i E) = \binom{d-g+1}{i} + \binom{d-g-1}{i} \binom{d-i-g}{2} + \binom{d-g-1}{i} \binom{i-g}{-}$$

Thus, we have

$$\begin{aligned}
 h^0(\Lambda^i E) &= \binom{d-g+1}{i} + \binom{d-g-1}{i-2} \binom{d-i-g}{-} \quad 2 \leq i \leq g-1, \\
 &+ \binom{d-g-1}{i-2} \binom{d-i-g}{-} \\
 &+ \binom{d-g-1}{i} \binom{i-g}{-}
 \end{aligned}$$

**PROPOSITION 3.5**

For any integer  $i \leq d - g$  satisfying  $(L^{-1} \otimes T^{i-1}) = 0$ , the map  $(L_i) \rightarrow (\Lambda^i E)$  induced by diagram (18) is injective. In particular, it is an isomorphism for  $i \leq g$ .

*Proof.* We get an exact commutative diagram

$$\begin{array}{ccccccc}
 0 & \rightarrow & \Gamma(L \otimes T^{-(d-g-1)+(i-1)}) \otimes \Lambda^{i-1} U^* & \rightarrow & \Gamma(\Lambda^i E) & \rightarrow & \Lambda^i U^* \otimes \Gamma(T^i) \rightarrow \dots \\
 & & \alpha_1 \uparrow & & \alpha_2 \uparrow & & \alpha_3 \uparrow \\
 0 & \longrightarrow & \Gamma(W^*(i-1)) \otimes \Lambda^{i-1} U^* & \longrightarrow & \Gamma(L'_i) & \rightarrow & \Lambda^i U^* \otimes \Gamma(\mathcal{O}(i)) \rightarrow \dots
 \end{array}$$

where  $\alpha_i$  s are induced by diagram (18).

- $\alpha_1$  is injective. Consider the left vertical sequence of (18),

$$0 \rightarrow L^{-1} \otimes \Lambda^{i-1} U^* \otimes T^{i-1} \rightarrow \pi^* W^* \otimes \Lambda^{i-1} U^* \rightarrow L \otimes T^{-(d-g-1)} \otimes \Lambda^{i-1} U^* \otimes T^{i-1} \rightarrow 0.$$

This gives rise to

$$0 \rightarrow (L^{-1} \otimes T^{i-1}) \otimes \Lambda^{i-1}U^* \rightarrow (W^*(i-1))_{a_1} (L \otimes T^{-(d-g-1)+(i-1)}) \otimes \Lambda^{i-1}U^* \rightarrow \dots$$

$$\rightarrow$$

Since

$$\begin{aligned} (W^* \otimes_{\mathbb{P}^1} O_{\mathbb{P}^1}(i-1)) &= (W \otimes_{\mathbb{P}^1} O_{\mathbb{P}^1}(d-g+i)) \\ &= * \otimes_{\mathbb{P}^1} O_{\mathbb{P}^1}(-g+i) \\ &\sim (\pi^* L \otimes_{\mathbb{P}^1} O_{\mathbb{P}^1}(d-g+i)) \\ &= * \otimes_{\mathbb{P}^1} O_{\mathbb{P}^1}(-g+i) \\ &\sim (\pi^* (L \otimes_{\mathbb{P}^1} \pi^* (O_{\mathbb{P}^1}(d-g+i)))) \\ &\sim (L \otimes_{\mathbb{P}^1} \pi^* (O_{\mathbb{P}^1}(d-g+i))) \\ &= \otimes_{\mathbb{P}^1} O_{\mathbb{P}^1}(-g+i) \\ &\sim (L \otimes_{\mathbb{P}^1} T^{-d+g+i}) \\ &= \otimes \end{aligned}$$

Thus  $a_1$  is injective.

- $a_3$  is injective by definition. Hence  $a_2$  is injective and since both the spaces have the

$$) \rightarrow (\Lambda^i E) \text{ is an isomorphism. same dimension for } i \leq g, \text{ the map } (L_i$$

#### 4. Construction of a subbundle of E

We want to prove the conjecture for hyperelliptic curves of genus  $g$ . We shall first do this for  $i = 2$ . The main point is to construct sufficiently many locally decomposable sections that are not globally decomposable.

Consider  $p : P(W^*) \rightarrow P^1$ , the natural projection. For every  $a \in P^1$ , the fibre  $l_a = p^{-1}(a)$  is a secant of the curve  $C$ . Let  $W_a^*$  be the fibre of  $W^*$  at  $a$ . Since  $W$  is globally

generated, we have  $(W)_{\mathbb{P}^1} \rightarrow W = 0$ . Hence  $W \otimes_{\mathbb{P}^1} \pi^* L$  and  $(\pi^* L) \sim (L)$  and thus we have  $(L)_{\mathbb{P}^1} \rightarrow W = 0$ , which gives  $W^* \cong (L^*)_{\mathbb{P}^1}$  and we can identify  $(L)^*$  with  $(E)$ . Also, we have

$$0 \rightarrow E^* \rightarrow (L)_C \rightarrow L \rightarrow 0,$$

which gives  $(L)^*_C \rightarrow E$ , i.e. a map  $(E)_C \rightarrow E$ . Thus we get a map  $(W_a^*)^*_C \rightarrow (E)_C \rightarrow E$  which is composite of the inclusion of  $W_a^*$  in  $(E)$  and the evaluation map.

Let  $F(a)$  be the subbundle of  $E$  generated by the image of  $W_a^*$ . A section of  $(E)$  is non-zero at every point of  $C$  if it corresponds to a point of  $P((L)^*)$  not on the curve

$C$ , while a section corresponding to a point say  $x \in C$  vanishes exactly at  $x$ . Hence the map  $W_a^* \rightarrow F(a)$  is an isomorphism outside  $C \cap l_a$  but has rank 1 over  $C \cap l_a$ . The induced map  $\Lambda^2 W_a^* \rightarrow \Lambda^2 F(a)$  has simple zeros exactly over  $C \cap l_a$ . Hence  $F(a)$  has rank 2 and  $\Lambda^2 F(a) \cong T$ . The vector bundle  $F(a)$  has  $W^*$  as its space of sections hand,  $\dim(F(a)) = \dim(T) = 2$ . Thus we i.e.  $\dim(F(a)) = 2$ . On the other hand,  $\dim(\Lambda^2 F(a)) = \dim(T) = 2$ .

get a 2-dimensional subspace of  $(\Lambda^2 E)$  consisting of locally decomposable sections of

$$g \leq i \leq d - g.$$

which only the 1-dimensional subspace  $\Lambda^2(F(a)) \subset (\Lambda^2 F(a))$  consists of globally decomposable sections.

The next step is to globalize this construction, i.e. to vary the point  $a$ . We consider the graph inclusion  $\subset C \times P^1$  given by the map  $\pi$ . This divisor belongs to the line bundle  $p_1^* T \otimes p_2^* O_{P^1}(1)$ , where  $p_1$  and  $p_2$  are the natural projections to  $C$ , resp.  $P^1$ . The direct image by  $p_2$  of the bundle morphism  $p_2^* W^* \rightarrow (E)_{C \times P^1}$  yields the map  $W^* \rightarrow (E)_{P^1}$ , and hence a map  $\Lambda^2 W^* \rightarrow \Lambda^2 (E)_{P^1}$ .

On the other hand, the bundle homomorphism  $p_2^*W^* \rightarrow (E)_{C \times P^1} \rightarrow p_1^*E$  fails to be injective precisely over  $\cdot$ . Thus, we get a morphism  $p_2^*(\Lambda^2 W^*) \otimes \mathcal{O}(\cdot) \rightarrow p_1^*(\Lambda^2 E)$ . Taking direct image by  $p_2$  gives a morphism  $\Lambda^2 W^* \otimes (T) \otimes \mathcal{O}(1) \rightarrow (\Lambda^2 E)_{P^1}$ . For every  $a \in P^1$  this induces a map  $(T) \rightarrow (\Lambda^2 E)_{P^1}$  and this gives exactly the space of locally decomposable sections described above.

Finally, we get a commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Lambda^2 W^* & \longrightarrow & \Lambda^2 W^* \otimes \Gamma(T) \otimes \mathcal{O}_{P^1}(1) & \longrightarrow & \Lambda^2 W^* \otimes \mathcal{O}_{P^1}(2) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \Lambda^2 \Gamma(E)_{P^1} & \longrightarrow & \Gamma(\Lambda^2 E)_{P^1} & \longrightarrow & D^2_{P^1} \longrightarrow 0
 \end{array} \tag{20}$$

where  $D^2_{P^1} := \begin{matrix} \Lambda^2 \\ (E) \\ \mathcal{O} \end{matrix}$  and the top horizontal row is the evaluation sequence for  $P^1$  (1), which is

$$0 \rightarrow \mathcal{O}_{P^1}(-1) \rightarrow (\mathcal{O}_{P^1}(1)) \otimes \mathcal{O}_{P^1} \rightarrow \mathcal{O}_{P^1}(1) \rightarrow 0.$$

Tensoring with  $\Lambda^2 W^* \otimes \mathcal{O}_{P^1}(1)$ , we get

$$0 \rightarrow \Lambda^2 W^* \rightarrow \Lambda^2 W^* \otimes (T) \otimes \mathcal{O}_{P^1}(1) \rightarrow \Lambda^2 W^* \otimes \mathcal{O}_{P^1}(2) \rightarrow 0.$$

We have to show that the locally decomposable sections constructed above together with  $\Lambda^2(E)$  generate  $(\Lambda^2 E)$ . For this, we consider the map  $\Lambda^2 W^* \otimes \mathcal{O}_{P^1}(2) \rightarrow D^2_{P^1}$ . We want to show that this map is injective (as a bundle map) and that the resulting rational curve in  $P(D^2)$  is the rational normal curve of degree  $d - g - 3$  (recall that  $\dim D^2 = d - g - 2$ ). This is sufficient since the rational normal curve of degree  $n$  in  $P^n$  spans  $P^n$ .

Our aim is to do this by entirely reducing the problem to computations on  $P^1$ , respectively  $P^1 \times P^1$ .

*Lemma 4.1* [5]. *Let  $\mathcal{O}_{P^1}(-n) \rightarrow (\mathcal{O}_{P^1}(n))^*_{P^1}$  be a non-zero  $Sl_2(\mathbb{C})$ -equivariant morphism. Then this morphism defines an embedding of  $P^1$  into  $P((\mathcal{O}_{P^1}(n))^*)$  as a rational normal curve of degree  $n$ .*

We return to the bundle  $W$ . Sequence (16) gives for  $i = 2$  the following sequence:

$$\begin{array}{ccccccc}
 0 & & \Lambda^2 W^* & & \Lambda^2 (W^*)^* & & L & & 0 \\
 & \rightarrow & \Lambda & & \Lambda P & & \rightarrow & 2 & \rightarrow
 \end{array} \tag{21}$$

Consider  $P^1 \times P^1$  together with projections  $q_1$  and  $q_2$  respectively. Taking pullback of (21) via  $q_1$  and  $q_2$ , we get a map  $q_2^* \Lambda^2 W^* \rightarrow q_1^* L_2$  that vanishes along the diagonal  $\subset P^1 \times P^1$ . Hence, we get a morphism  $q_2^* \Lambda^2 W^* \otimes \mathcal{O}(\cdot) \rightarrow q_1^* L_2$ .

Applying  $q_{2*}$ , we get a map

$$\Lambda^2 W^* \otimes (\mathcal{O}(1)) \otimes \mathcal{O}(1) \rightarrow (L_2)_{P^1}.$$

This gives rise to a commutative diagram

$$\begin{array}{ccc}
 0 & & 0 \\
 \downarrow & & \downarrow \\
 \Lambda^2 W^* & \longrightarrow & \Lambda^2 \Gamma(W^*)^*_{P^1} \\
 \downarrow & & \downarrow \\
 \Lambda^2 W^* \otimes \Gamma(\mathcal{O}(1)) \otimes \mathcal{O}(1) & \longrightarrow & \Gamma(L'_2)_{P^1} \\
 \downarrow & & \downarrow \\
 \Lambda^2 W^* \otimes \mathcal{O}(2) & \longrightarrow & H^1(\Lambda^2 W^*) = \Gamma(\Lambda^2 W \otimes \mathcal{O}(-2))^* \\
 \downarrow & & \downarrow \\
 0 & & 0
 \end{array} \tag{22}$$

where the left-hand column is the Euler sequence on  $\mathbb{P}^1$  twisted by  $\Lambda^2 W^* \otimes \mathcal{O}(1)$ , the right hand column comes from (21) and the map  $\Lambda^2 W^* \rightarrow \Lambda^2(W)^*$  is the natural one. This diagram is  $Sl_2(\mathbb{C})$  equivariant, where  $Sl_2(\mathbb{C})$  acts on  $\mathbb{P}^1$  in the usual way and on  $\mathbb{P}^1 \times \mathbb{P}^1$  by the diagonal action. In particular the morphism  $\Lambda^2 W^* \otimes \mathcal{O}(2) \rightarrow (\Lambda^2 W \otimes \mathcal{O}(-2))^*$  is  $Sl_2(\mathbb{C})$  equivariant, by Lemma 4.2, it defines an embedding of  $\mathbb{P}^1$  into  $\mathbb{P}(\Lambda^2 W \otimes \mathcal{O}(-2))^*$  as a rational normal curve of degree  $d - g - 3$ .

Lemma 4.2 [5]. Diagram (22) gives rise to a commutative and exact diagram

$$\begin{array}{ccccccc}
 0 & \rightarrow & \Gamma(\Lambda^2 W \otimes \mathcal{O}(-2)) & \longrightarrow & \Gamma(I'_2)^* & \longrightarrow & \Lambda^2 \Gamma(W) \rightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \rightarrow & \Gamma(\Lambda^2 W \otimes \mathcal{O}(-2)) & \rightarrow & \Gamma(\Lambda^2 W \otimes \mathcal{O}(-1)) \otimes \Gamma(\mathcal{O}(1)) & \rightarrow & \Gamma(\Lambda^2 W) \rightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array} \tag{23}$$

PROPOSITION 4.3

$(\Lambda^2 E)$  is generated by locally decomposable sections.

Proof. We have constructed maps  $p_2^*(\Lambda^2 W^*) \otimes \mathcal{O}(\ ) \rightarrow p_1^*(\Lambda^2 E)$  on  $C \times \mathbb{P}^1$  and  $q_2^* \Lambda^2 W^* \otimes \mathcal{O}(\ ) \rightarrow q_1^* L_2$  on  $\mathbb{P}^1 \times \mathbb{P}^1$ . Consider the diagram

$$\begin{array}{ccc}
 C & \xleftarrow{p_1} & C \times \mathbb{P}^1 & \xrightarrow{p_2} & \mathbb{P}^1 \\
 \downarrow \pi & & \downarrow \pi \times id & & \parallel \\
 \mathbb{P}^1 & \xleftarrow{q_1} & \mathbb{P}^1 & \xrightarrow{q_2} & \mathbb{P}^1 \\
 & & \times & & 
 \end{array} \tag{24}$$

Pulling the morphism  $q_2^* \Lambda^2 W^*$

morphism  $p_2^*(\Lambda^2 W^*) \otimes \mathcal{O}(\ )$

$$p_2^*(\Lambda^2 W^*) \otimes \mathcal{O}(\Gamma)$$

$$\parallel$$

$$p_2^*(\Lambda^2 W^*) \otimes \mathcal{O}(\Gamma)$$

$$\begin{array}{l}
 \rightarrow q_1^* L_2 \text{ on } \mathbb{P}^1 \times \mathbb{P}^1 \text{ back to } C \times \mathbb{P}^1, \text{ we get a } \rightarrow p_1^*(\pi \\
 \text{struction the diagram,} \\
 \rightarrow p_1^*(\pi^* L'_2) \\
 \downarrow \\
 p_1^*(\Lambda^2 E)
 \end{array} \tag{25}$$

commutes where the map  $p_1^*(\pi^* L_2) \rightarrow p_1^* \Lambda^2 E$  is the pullback via  $p_1$  of the corresponding map in diagram (18). Pushing this down via  $\pi \times id$  to  $\mathbb{P}^1 \times \mathbb{P}^1$  leads to the commutative diagram

$$\begin{array}{ccc}
 q_2^* \Lambda^2 W^* \otimes \mathcal{O}(\Delta) & \longrightarrow & q_1^* L'_2 \\
 \parallel & & \downarrow \\
 q_2^* \Lambda^2 W^* \otimes \mathcal{O}(\Delta) & \rightarrow & q_2^* \Lambda^2 W^* \otimes (\pi \times id)_* \mathcal{O}(\Gamma) \rightarrow q_1^*(\pi_* \pi^* L'_2) \\
 \parallel & & \parallel & & \downarrow \\
 q_2^* \Lambda^2 W^* \otimes \mathcal{O}(\Delta) & \rightarrow & q_2^* \Lambda^2 W^* \otimes (\pi \times id)_* \mathcal{O}(\Gamma) \rightarrow q_1^*(\pi_* \Lambda^2 E)
 \end{array} \tag{26}$$

Now taking  $q_2^*$  of the outermost square, we get

$$\begin{array}{ccc}
 \Lambda^2 W^* \otimes \Gamma(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_{\mathbb{P}^1}(1) & \rightarrow & \Gamma(\mathbb{P}^1, L'_2) \\
 \parallel & & \downarrow \\
 \Lambda^2 W^* \otimes \Gamma(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_{\mathbb{P}^1}(1) & \rightarrow & \Gamma(C, \Lambda^2 E)
 \end{array} \tag{27}$$

where the right-hand vertical map is an isomorphism from Proposition 4.3. Thus in order to compute the diagram

$$\begin{array}{ccc}
 \Lambda^2 W^* & \longrightarrow & \Lambda^2 \Gamma(W)^*_{\mathbb{P}^1} \\
 \downarrow & & \downarrow \\
 \Lambda^2 W^* \otimes \Gamma(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_{\mathbb{P}^1}(1) & \longrightarrow & \Gamma(\Lambda^2 E)
 \end{array} \tag{28}$$

we can compute

$$\begin{array}{ccc}
 \Lambda^2 W^* & \longrightarrow & \Lambda^2 \Gamma(W)^*_{\mathbb{P}^1} \\
 \downarrow & & \downarrow \\
 \Lambda^2 W^* \otimes \Gamma(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_{\mathbb{P}^1}(1) & \longrightarrow & \Gamma(L'_2)_{\mathbb{P}^1}
 \end{array} \tag{29}$$

and the result follows from Lemma 4.2.

**5. Proof of the main result**

Here we prove Theorem 1. We shall first show that for  $2 \leq i \leq d - g$ , there is a natural epimorphism

$$\Lambda^{i-2} (W)^* \otimes (L_2)_{\mathbb{P}^1} \rightarrow (L_i)_{\mathbb{P}^1} \rightarrow 0.$$

Setting  $i = 2$  in (16), we get

$$0 \rightarrow \Lambda^2 W^* \rightarrow \Lambda^2 (L)_{\mathbb{P}^1} \rightarrow L_2 \rightarrow 0.$$

Twisting with  $\Lambda^{i-2} (W)^*$ , we get an exact sequence

$$0 \rightarrow \Lambda^{i-2} (W)^* \otimes \Lambda^2 W^* \rightarrow \Lambda^{i-2} (W)^* \otimes \Lambda^2 (W)^* \rightarrow \Lambda^{i-2} (W)^* \otimes L_2 \rightarrow 0$$

(since  $(L) = (W)$ ) Combining this with (16), we get

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Lambda^{i-2} \Gamma(W)^* \otimes \Lambda^2 W^* & \longrightarrow & \Lambda^{i-2} \Gamma(W)^* \otimes \Lambda^2 \Gamma(W)^* & \rightarrow & \Lambda^{i-2} \Gamma(W)^* \otimes L'_2 \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & \Lambda^{i-2} \Gamma(W(-1))^* \otimes \mathcal{O}(i-2) \otimes \Lambda^2 W^* & \longrightarrow & \Lambda^i \Gamma(W)^* & \longrightarrow & L'_i \longrightarrow 0
 \end{array} \tag{30}$$

Here the middle vertical map is the canonical one and the left hand vertical map is given by taking  $\Lambda^{i-2}$  of the dual evaluation sequence

$$0 \rightarrow W^* \rightarrow (W)^* \otimes \mathcal{O}_{\mathbb{P}^1} \rightarrow (W(-1))^* \otimes \mathcal{O}_{\mathbb{P}^1}(1) \rightarrow 0.$$

Taking  $\Lambda^{i-2}$  of the above sequence, we get

$$\begin{array}{l}
 0 \rightarrow \Lambda^2 W^* \otimes \Lambda^{i-2} (W(-1))^* \otimes \mathcal{O}_{\mathbb{P}^1}(i-2) \rightarrow \Lambda^{i-2} (W)^* \rightarrow F_{i-2} \rightarrow 0, \\
 0 \rightarrow F_{i-2} \rightarrow \Lambda^{i-2} (W)^* \otimes \mathcal{O}_{\mathbb{P}^1} \rightarrow \Lambda^{i-2} (W(-1))^* \otimes \mathcal{O}_{\mathbb{P}^1}(i-2) \rightarrow 0.
 \end{array}$$

$$0 \rightarrow F_{i-2} \otimes \Lambda^2 W^* \rightarrow \Lambda^{i-2} (W)^* \otimes \mathcal{O}_{\mathbb{P}^1} \otimes \Lambda^2 W^* \rightarrow \Lambda^{i-2} (W(-1))^* \otimes \mathcal{O}_{\mathbb{P}^1}(i-2) \otimes \Lambda^2 W^* \rightarrow 0.$$

Taking the associated cohomology sequence of (30), we get the following commutative diagram:

$$\begin{array}{ccc}
 0 & & 0 \\
 \downarrow & & \downarrow \\
 \Lambda^{i-2} \Gamma(W)^* \otimes \Lambda^2 \Gamma(W)^* & \longrightarrow & \Lambda^i \Gamma(W)^* \\
 \downarrow & & \downarrow \\
 \Lambda^{i-2} \Gamma(W)^* \otimes \Gamma(L'_2) & \longrightarrow & \Gamma(L'_i) \\
 \downarrow & & \downarrow \\
 H^1(\Lambda^{i-2} \Gamma(W)^* \otimes \Lambda^2 W^*) & \rightarrow & H^1(\Lambda^{i-2} \Gamma(W(-1))^* \otimes \mathcal{O}(i-2) \otimes \Lambda^2 W^*) \\
 \downarrow & & \downarrow \\
 0 & & \vdots
 \end{array} \tag{31}$$

Here  $W$  is a rank 2 vector bundle on  $\mathbb{P}^1$  of degree  $d - g - 1$ .

$$\begin{array}{l}
 = \mathcal{O} \quad - \quad - \\
 \det W \sim (d \quad g \quad 1), \\
 \det W^* \sim (d \quad - \quad 1), \\
 = \mathcal{O} \quad - \quad + \quad +
 \end{array}$$

i.e.

$$\Lambda^2 W^* \sim \binom{d-g-1}{O} + \binom{g-1}{+} + \binom{1}{+}$$

Since  $2 \leq d - g - 1$ . Therefore  $(\Lambda^2 W) = 0$ ,

The top horizontal map is clearly surjective. The bottom horizontal map is surjective since  $H^2$  vanishes on  $P^1$ . By standard diagram chasing the middle horizontal map must be surjective thus giving our first claim.

By construction the natural diagram

$$\begin{array}{ccc} \Lambda^{i-2}\Gamma(W)^* \otimes \Gamma(L'_2) & \rightarrow & \Lambda^{i-2}\Gamma(E) \otimes \Gamma(\Lambda^2 E) \\ \downarrow & & \downarrow \\ \Gamma(L'_i) & \xrightarrow{\quad} & \Gamma(\Lambda^i E) \end{array} \tag{32}$$

commutes.

Since  $(W)^* \sim (L)^*$  and  $(L)^*$  can be identified with  $(E)$ , by Proposition 3.5, the horizontal maps are isomorphisms for  $i \leq g$ . Hence the natural map  $\Lambda^{i-2}(E) \otimes (\Lambda^2 E) \rightarrow (\Lambda^i E)$  is surjective for  $i \leq g$  and our claim follows from Proposition 4.3.

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